1. Recall that

$$\|f\|_u = \sup\{|f(x)| \mid x \in S\}$$

(a) By triangle inequality we have

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|$$

Taking supremum both sides and using

$$\sup_{x \in S}(|f(x)| + |g(x)|) \leq \sup_{x \in S}|f(x)| + \sup_{x \in S}|g(x)|$$

gives the inequality.

(b) Clearly

$$|f(x)g(x)| = |f(x)||g(x)|$$

Taking supremum both sides and using

$$\sup_{x \in S}|f(x)||g(x)| \leq \left(\sup_{x \in S}|f(x)|\right)\left(\sup_{x \in S}|g(x)|\right)$$

(since $|f|$ and $|g|$ are nonnegative) gives the inequality.

2. (a) Let $\varepsilon > 0$ and $x \in S$. Since $f_n \to f$ pointwise there is $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|f_n(x) - f(x)| < \varepsilon/2$. Similarly there is $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|g_n(x) - g(x)| < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$ and $n > N$ implies

$$|f_n(x) + g_n(x) - f(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon$$

so $f_n + g_n = h_n \to f + g$ pointwise.

(b) The exact same argument as above works. Instead one does not choose a specific $x \in S$ at the first place to account for uniform convergence.

3. Suppose for a contradiction that there is $x, y \in (a, b)$ with $x < y$ such that $f(y) < f(x)$. Let $\varepsilon = f(x) - f(y) > 0$. Since $f_n \to f$ pointwise, there is $N_1$ such that $n > N_1$ implies

$$|f_n(x) - f(x)| < \varepsilon/3 \implies f_n(x) > f(x) - \varepsilon/3$$

Similarly there is $N_2$ such that $n > N_2$ implies

$$|f_n(y) - f(y)| < \varepsilon/3 \implies f_n(y) < f(y) + \varepsilon/3$$

Let $N = \max\{N_1, N_2\}$ and $n > N$ implies that

$$f_n(y) - f_n(x) < f(y) + \varepsilon/3 - f(x) + \varepsilon/3 = -\varepsilon + 2\varepsilon/3 = -\varepsilon/3 < 0$$

a contradiction since $f_n$ is non-decreasing.

4. Let $\varepsilon > 0$. Since $f$ is uniformly continuous, there is $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Choose $N$ such that $\frac{1}{N} < \delta$, then $n > N$ implies

$$|f_n(x) - f(x)| = |f(x + 1/n) - f(x)| < \varepsilon$$

so $f_n \to f$ uniformly.

5. (a) Since $h$ is continuous on $[a, b]$, $h$ is uniformly continuous on $[a, b]$ (bounded interval implies uniform continuity), therefore $f$ is also uniformly continuous (since it is constant outside of $[a, b]$). $f$ is clearly bounded.
(b) By the fundamental theorem of calculus we have
\[
f'_n(x) = \frac{n}{2} (f(x + 1/n) - f(x - 1/n))
\]
which is uniformly continuous (since composition of uniformly continuous functions is still uniformly continuous). Moreover,
\[
|f_n(x)| \leq \frac{n}{2} (x + 1/n - (x - 1/n)) \sup_{x \in (x-1/n,x+1/n)} |f(x)| \leq \|f\|_u
\]
So \(\|f_n\|_u \leq \|f\|_u\).
(c) Let \(\varepsilon > 0\). Since \(f\) is uniformly continuous, there is \(\delta > 0\) such that \(|x - y| < \delta\) implies \(|f(x) - f(y)| < \varepsilon\). Choose \(N\) such that \(1/N < \delta\). For \(n > N\) we have
\[
|f_n(x) - f(x)| = \frac{n}{2} \left| \int_{x-1/n}^{x+1/n} f(t)dt - \int_{x-1/n}^{x+1/n} f(x)dt \right| \leq \frac{n}{2} \left| \int_{x-1/n}^{x+1/n} |f(x) - f(t)| dt \right|
\]
which is uniformly continuous (since composition of uniformly continuous functions is still uniformly continuous).

(b) By linearity of the limit it suffices to show for \(x \in \mathbb{R}\),
\[
\lim_{n \to \infty} \frac{x^{-(k+1)}e^{-1/x}}{e^{1/x}} = \lim_{n \to \infty} \frac{x^{-k}e^{-1/x}}{e^{1/x}} = \lim_{x \to 0} \frac{(k+1)x^{-k}}{e^{1/x}} = 0
\]
Hence by induction we have \(\lim_{x \to 0} x^{-n}e^{-1/x} = 0\) for all \(n \in \mathbb{N}\). Thus \(f(x)\) is continuous.

6. (a) The function clearly has no issues for \(x \neq 0\). We need to check that \(\lim_{x \to 0} P(1/x)\phi(x) = 0\) for any polynomial \(P\). By linearity of the limit it suffices to show for \(P(x) = x^n\) for \(n \in \mathbb{N}\) we have
\[
\lim_{x \to 0} P(1/x)\phi(x) = 0
\]
We use induction. This is clearly true for \(n = 0\). Suppose the claim is true for all \(n = k\), then for \(n = k + 1\) by L'Hopital's rule we have
\[
\lim_{x \to 0} x^{-(k+1)}e^{-1/x} = \lim_{x \to 0} \frac{x^{-(k+1)}e^{-1/x}}{e^{-x^2}} = \lim_{x \to 0} \frac{(k+1)x^{-k}}{e^{1/x}} = 0
\]
(b) We use induction again. For \(k = 0\) the statement is clearly true. Suppose \(\phi^{(k)}(x) = P_k(1/x)\phi(x)\) for some polynomial \(P_k\). For \(x > 0\) using the chain rule we have
\[
\phi^{(k+1)}(x) = P'_k(1/x) \cdot (-1/x^2)\phi(x) + \left( \frac{1}{x^2} P_k(1/x) \right)\phi(x) = ((-x^2P'_k(1/x) + (x^2P_k(1/x)))\phi(x)
\]
which is again of the form \(P_{k+1}(1/x)\phi(x)\), where
\[
P_{k+1}(x) = -x^2P'_k + x^2P_k
\]
is again a polynomial. For \(x < 0\) clearly \(\phi^{(k+1)}(x) = P_{k+1}(1/x)\phi(x)\) still holds since \(\phi(x)\) is identically 0. Finally we have
\[
\lim_{x \to 0} \phi^{(k+1)}(x) = 0
\]
by part (a). So \(\phi^{(k+1)}(x)\) is a continuous function of the form \(\phi^{(k+1)}(x) = P_{k+1}(1/x)\phi(x)\) for all \(x \in \mathbb{R}\). Consequently it is a smooth function on \(\mathbb{R}\) by part (a).

7. (a) Consider \(\psi(x) = \phi((x-a)(b-x))\) for \(x \in (a, b)\). Clearly when \(x \leq a\) or \(x \geq b\) we have \(\psi(x) = 0\), and when \(x \in (a, b)\) we have \(\psi(x) > 0\) hence \((x-a)(b-x) > 0\). Moreover \(\psi\) is smooth since \(\phi\) is.
(b) Let
\[
C = \int_a^b \psi(x)dx
\]
(which can be computed explicitly). Now define
\[
\eta(x) = \frac{1}{C} \int_{-\infty}^x \psi(t)dt
\]
Clearly \( \eta \) is again a smooth function. Moreover when \( x < a \), \( \eta(x) = 0 \) since \( \psi(x) = 0 \), and when \( x > b \) we have

\[
\eta(x) = \frac{1}{C} \int_{-\infty}^{b} \psi(t)dt = \frac{1}{C} \int_{a}^{b} \psi(t)dt = 1
\]

Finally when \( x \in (a, b) \) we must have \( 0 \leq \eta(x) \leq 1 \) since \( \eta(x) \) is clearly increasing.

(c) Let \( \psi_1(x) \) be the function as constructed in part (b) such that \( \psi_1(x) = 0 \) for \( x \leq a \) and \( x = 1 \) for \( x \geq c \). Let \( \psi_2(x) \) be the function as constructed in part (b) such that \( \psi_2(x) = 0 \) for \( x \leq d \) and \( \psi_2(x) = 1 \) for \( x \geq b \). Consider \( \zeta(x) = \psi_1(x) - \psi_2(x) \) which is clearly smooth. For \( x \leq a \) we have \( \psi_1(x) = \psi_2(x) = 0 \). For \( a \leq x \leq c \) we have \( \zeta(x) = \psi_1(x) \) which is between 0 and 1. For \( c \leq x \leq d \) we again have \( \zeta(x) = \psi_1(x) \) which is identically 1. For \( d \leq x \leq b \) we have \( \zeta(x) = 1 - \psi_2(x) \) which is again between 0 and 1. Finally for \( x \geq b \) we have \( \zeta(x) = 1 - 1 = 0 \) again. So this is precisely what we want.