Solutions Final Exam — December 11, 2019

1. (a) (10 points) State the Min/Max Theorem (also called the Extreme Value Theorem).

> The Min/Max theorem states that if $f : [a, b] \to \mathbb{R}$ is continuous, then there are $c, d \in [a, b]$ so that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$. That is, f achieves both its maximum and minimum value on the closed bounded interval $[a, b]$.

(b) (5 points) Give an example of a continuous function $f : (-1, 1) \to \mathbb{R}$ that achieves its maximum value, but does not achieve its minimum value.

> An example would be $f(x) = 1 - x^2$. This function achieves its maximum value of $1 = f(0)$ at $x = 0$ but does not achieve a minimum value. This is consistent with a) as the interval is not closed

(c) (15 points) Show that if $f : [0, 1] \to (0, 1)$ is continuous, then f is *not* onto.

As f is continuous and $[0, 1]$ is a closed bounded interval, the Min/Max Theorem implies that there is a value $c \in [0, 1]$ and a value $d \in [0, 1]$ so that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$. That is, $f([0, 1]) \subset [f(c), f(d)] \subset (0, 1)$. By definition, one has $0 < f(c) \leq f(d) < 1$ and so there is some element $y \in (0, f(c))$. However, there can be no $x \in [0, 1]$ so $f(x) = y$ as one must have $f(x) \geq f(c) > y$. Hence, f cannot be onto.

2. (a) (10 points) State the formal definition of uniform continuity of a function $f : (a, b) \to \mathbb{R}$.

The given function f is uniformly continuous if, for all $\epsilon > 0$, there is a $\delta > 0$ so that if $x, y \in (a, b)$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

(b) (5 points) Give an example of a continuous function $f : (a, b) \to \mathbb{R}$ that is not bounded. You do not need to justify your answer.

An example with $(a, b) = (0, 1)$ is $f(x) = \frac{1}{x}$.

(c) (15 points) Show that if $f : (a, b) \to \mathbb{R}$ is uniformly continuous, then there is a number M so $|f(x)| \leq M$ for all $x \in (a, b)$. That is, f is bounded.

As f is uniformly continuous, there is a $\delta > 0$ so that $x, y \in (a, b)$ with $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$. Choose, $N \in \mathbb{N}$ so $\delta N > (b - a)$ and let $x_i = a + \frac{b-a}{N+1} i$. Observe $x_i \in (a, b)$ for $1 \leq i \leq N$ and $x_{i+1} - x_i = \frac{b-a}{N+1} < \delta$. In particular, for every $x \in (a, b)$ there is a $1 \leq j \leq N$ so $|x - x_j| < \delta$. Set $M = \max\{|f(x_1)|, \ldots, |f(x_N)|\} + 1$.
As already observed, for any $x \in (a, b)$ there is a $1 \leq j \leq N$ so $|x - x_j| < \delta$. Using the reverse triangle inequality this gives

$$|f(x)| = |f(x) - f(x_j) + f(x_j)| \leq |f(x) - f(x_j)| + |f(x_j)| \leq 1 + |f(x_j)| \leq M$$

This proves the claim.

3. (a) (5 points) State the definition of a function $f : (a, b) \to \mathbb{R}$ being strictly increasing on (a, b).

The function f is strictly increasing if $x, y \in (a, b)$ with $x < y$ satisfies $f(x) < f(y)$.

(b) (10 points) Show that if $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b).

Pick $x, y \in (a, b)$ with $x < y$. By the mean value theorem applied to $[x, y] \subset (a, b)$, there is a $c \in (x, y)$ so that $f(y) - f(x) = f'(c)(y - x)$. As $f'(c) > 0$ and $y - x > 0$ one concludes that $f(y) - f(x) > 0$. That is $f(y) > f(x)$. This means that f is strictly increasing.

(c) (10 points) Show that if $f : (a, b) \to \mathbb{R}$ is strictly increasing and $c \in (a, b)$, then $\lim_{x \to c_-} f(x)$ and $\lim_{x \to c_+} f(x)$ both exist and satisfy $\lim_{x \to c_-} f(x) \leq f(c) \leq \lim_{x \to c_+} f(x)$.

Set $S_- = \{f(x) : x \in (a, c)\}$. We observe that S_- is a non-empty set with upper bound $f(c)$. This is because there are $y \in (a, c)$ and the monotonicity of f implies $f(y) < f(c)$. By the least upper bound property of $\mathbb{R}$, this means there is a value $L_- = \sup S_- \leq f(c)$. We claim $\lim_{x \to c^-} f(x) = L_-$. Indeed, given $\epsilon > 0$, the definition of least upper bound implies there is a $y \in S_-$ with $f(y) > L_- - \epsilon$. The fact that f is strictly increasing implies that for $x \in (y, c)$ one has $L_- - \epsilon < f(y) < f(x) < f(c)$. That is if we set $\delta = c - y > 0$ then for any x so that $x < c$ and $0 < |x - c| < \delta$ one has $|f(x) - L_-| < \epsilon$. That means $\lim_{x \to c^-} f(x) = L_- \leq f(c)$. A similar argument proves that $\lim_{x \to c^+} f(x)$ exists and is greater than or equal to $f(c)$.

(d) (10 points) Show that if $g : (a, b) \to \mathbb{R}$ is differentiable and $g' : (a, b) \to \mathbb{R}$ is strictly increasing, then g' is continuous. (Hint: Recall, the derivative of a differentiable function has the intermediate value property).

By a theorem of Darboux, g' has the intermediate value property on any interval $[a', b'] \subset (a, b)$. Given a $c \in (a, b)$ observe that, by the previous problem $\lim_{x \to c^-} g'(x) = L_-$ and $\lim_{x \to c^+} g'(x) = L_+$ both exist and satisfy $\lim_{x \to c^-} g'(x) \leq f(c) \leq \lim_{x \to c^+} g'(x)$. We claim $\lim_{x \to c^-} g'(x) = g'(c) = \lim_{x \to c^+} g'(x)$. To see this observe, that if $L_- < g'(c)$, then any $d \in (a, c)$, and $y \in (L_-, g'(c))$ the intermediate value property of g' applied to $[d, c]$ implies there is an $x \in (d, c)$ so $g'(x) = y$. However, as g' is strictly increasing and $a < x < c$ one must have $g'(x) \leq L_-$. This contradicts $g'(x) > L_-$ and so one must have $L_- = g'(c)$. A similar argument shows $L_+ = g'(c)$.

Hence, $\lim_{x \to c} g'(x)$ exists and equals $g'(c)$. That is, g' is continuous at c and so g' is a continuous function.

4. (a) (10 points) Show that if $f : [a, b] \to [0, 1]$ satisfies $f(x) = 0$ for all $x \in [a, b] \cap \mathbb{Q}$, then

$$\underline{\int_a^b} f(x)dx = 0.$$

That is the lower Darboux integral of f vanishes.

For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, we observe that as $f(x) \geq 0$ one has $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \geq 0$. However, as the rationals are dense, one must have $[x_{i-1} \cap x_i] \cap \mathbb{Q} \neq \emptyset$. In particular, there is a $y \in [x_{i-1}, x_i]$ so $f(y) = 0$. It follows that $m_i \leq 0$ and so $m_i = 0$. Hence,

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta_i = 0$$

and so

$$\underline{\int_a^b} f(x)dx = \sup \{L(P, f) : P \text{ a partition}\} = 0.$$

(b) (5 points) Give an example of a discontinuous function $f : [0, 1] \to \mathbb{R}$ that is Riemann integrable.

The function $f(x) = \begin{cases} 1 & x \in [0, 1) \\ 0 & x = 1 \end{cases}$ is discontinuous at $x = 1$. However, this is the only discontinuity and so f is Riemann integrable.

(c) (20 points) Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous. Show directly from definitions that if $g : [a, b] \to \mathbb{R}$ satisfies $g(x) = f(x)$ for $x \in (a, b)$, then g is Riemann integrable.

As f is uniformly continuous on the interval (a, b), it is bounded by some value M_0 as shown (for instance) in a previous question. Set $M = \max\{M_0, |g(a)|, |g(b)|\}$ one clearly has $|g(x)| \leq M$ for all $x \in [a, b]$ and so g is bounded. In particular, it is enough to show that

$$\overline{\int_a^b} g(x)dx = \underline{\int_a^b} g(x)dx$$

That is, to show for any $\epsilon > 0$ that there is a partition $P = P_\epsilon$ so that

$$0 \leq \overline{\int_a^b} g(x)dx - \underline{\int_a^b} g(x)dx \leq U(P, f) - L(P, f) \leq \epsilon.$$

To that end, observe that, as f is uniformly continuous, for any $\epsilon > 0$, there is a $\delta > 0$ so that $x, y \in (a, b)$ with $|x - y| < \delta$ implies

$$|f(x) - f(y)| < \frac{\epsilon}{b - a}.$$

Now pick a partition $P = \{a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b\}$ of $[a, b]$ chosen so $\Delta_i < \min\left\{\delta, \frac{\epsilon}{4M}\right\}$. Set $M_i = \sup_{x\in[x_{i-1},x_i]} g(x)$ and $m_i = \inf_{x\in[x_{i-1},x_i]} g(x)$. For $2 \leq i \leq n$, the definition of f ensures that $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$ and $m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$ and so, by the uniform continuity of f and fact that $x_i - x_{i-1} < \delta$, when $2 \leq i \leq n$ one has $M_i - m_i \leq \frac{\epsilon}{b-a}$. One readily uses the bound on g to see that $M_1 - m_1 \leq 2M$ and $M_{n+1} - M_{n+1} \leq 2M$. Hence,

$$U(P, f) - L(P, f) = \sum_{i=1}^{n+1} (M_i - m_i)\Delta_i \leq 2M\Delta_1 + 2M\Delta_{n+1} + \frac{\epsilon}{b - a}(x_n - x_1) \leq \epsilon.$$

As ϵ is arbitrary, the claim is proved.

5. (a) (15 points) State both versions of the Fundamental Theorem of Calculus.

The first version of the fundamental theorem of calculus states that if$F : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b) and there is a $f : [a, b] \to \mathbb{R}$ that is Riemann integrable and so $F'(x) = f(x)$ for all $x \in (a, b)$, then

$$F(b) - F(a) = \int_a^b f(x)dx.$$

The second version states that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $F(x) = \int_a^x f(t)dt$, then F is continuous. Moreover, if f is continuous at $c \in (a, b)$, then F is differentiable at c and $F'(c) = f(c)$.

(b) (5 points) Give an example of a function $f : [-1, 1] \to \mathbb{R}$ that is Riemann integrable, but $F(x) = \int_0^x f(t)dt$ is not differentiable at $x = 0$.

Consider the function

$$f(x) = \begin{cases} -1 & x \in [-1, 0] \\ 1 & x \in (0, 1] \end{cases}$$

This function has only one discontinuity (at $x = 0$) so is Riemann integrable. One verfies that $F(x) = |x|$ and so F is not differentiable at $x = 0$.

(c) (15 points) Show that if $f : (-1, 1) \to \mathbb{R}$ is C^1 with $f(0) = 0$ and $f'(x) \geq 2|x|$, then $|f(x)| \geq x^2$ for all $x \in (-1, 1)$.

First observe that there is nothing to show when $x = 0$. We treat two case $x \in (0, 1)$ and $x \in (-1, 0)$. In the first case, the hypotheses on f (namely that it is C^1) allow us to apply the first version of the fundamental theorem of calculus to conclude that

$$|f(x)| \geq f(x) = f(x) - f(0) = \int_0^x f'(t)dt \geq \int_0^x 2t = \int_0^2 \frac{d}{dt}(t^2)dt = x^2.$$

In a similar, fashion, when $x \in (-2, 0)$ one has

$$|f(x)| \geq -f(x) = f(0) - f(x) = \int_x^0 f'(t)dt \geq \int_x^0 (-2t)dt = \int_x^0 \frac{d}{dt}(-t^2)dt = x^2.$$

Putting these together prove te claim.

6. (a) (10 points) State the definition of a sequence of functions $f_n : [a, b] \to \mathbb{R}$ uniformly converging to $f : [a, b] \to \mathbb{R}$.

The sequence converges uniformly on $[a, b]$ if, for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that if $n \geq N$, then $\sup_{x\in[a,b]} |f(x) - f_n(x)| < \epsilon$.

(b) (5 points) Give an example of a sequence of functions $f_n : [0, 1] \to \mathbb{R}$ so that f_n converges pointwise to $f : [0, 1] \to \mathbb{R}$ but not uniformly.

An example is given by $f_n(x) = x^n$. One has $\lim_{n\to\infty} x^n = 0$ if $x \in [0, 1)$ while $\lim_{n\to\infty} x^n = 1$ for $x = 1$ and so f_n converge pointwise to the function

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$$

However, the convergence cannot be uniform (as for instance the uniform limit of continuous functions is continuous and f is not.)

(c) (20 points) Prove that if $f_n : [a, b] \to \mathbb{R}$ are continuous and the f_n converge uniformly to $f : [a, b] \to \mathbb{R}$, then f is continuous.

Fix an $\epsilon > 0$. As $f_n \to f$ uniformly, there is an $N \in \mathbb{N}$ so that for $n \geq N$, $\sup_{x\in[a,b]} |f(x) - f_n(x)| < \epsilon/3$. Now fix an $c \in [a, b]$. As f_N is continuous, there is a $\delta > 0$ so for all $x \in [a, b]$ with $|x - c| < \delta$ one has $|f_N(x) - f_N(c)| < \epsilon/3$. Using the triangle inequality, one then concludes that, for $x \in [a, b]$ with $|x - c| < \delta$, one has

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

That is, f is continuous at $x = c$. As c is arbitrary, one concludes that f is continuous.