Solutions Final Exam — May. 14, 2014

1. Determine whether the following statements are true or false. Justify your answer (i.e., prove the claim, derive a contradiction or give a counter-example).

   (a) (10 points) There exist open intervals I_n with $I_{n+1} \subset I_n$ so that $\cap_{n=1}^{\infty} I_n = \emptyset$.

   > True. Let $I_n = (0, 1/n)$. If $z \in \cap_{n=1}^{\infty} I_n$, then $0 < z < \frac{1}{n}$ for all n, which violates the Archimedean principle.

   (b) (10 points) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and $\{x_n\}$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

   > True. Given $\epsilon > 0$, use the uniform continuity of f to pick $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Now use the Cauchy property of $\{x_n\}$ to pick an N so that if $N < m, n$, then $|x_m - x_n| < \delta$. Hence, if $N < m, n$, then $|f(x_n) - f(x_m)| < \epsilon$, i.e., $\{f(x_n)\}$ is Cauchy.

(c) (10 points) If $f : (a, b) \to \mathbb{R}$ is C^1 and strictly increasing, then $f'(x) > 0$ for each $x \in (a, b)$.

False. Let $(a, b) = (-1, 1)$ and $f(x) = x^3$, then $x < y$ implies $f(x) < f(y)$, but $f'(0) = 0$.

(d) (10 points) If $f : (-1, 1) \to \mathbb{R}$ is C^2 with $f(0) = f'(0) = 0$ and $f''(0) = 2$, then there is an interval I containing 0 so that $f(x) \geq 0$ for $x \in I$.

True. By Taylor's theorem $f(x) = \frac{1}{2}f''(0)x^2 + o(x^2) = x^2 + o(x^2)$. Choose, $\epsilon > 0$ so that if $|x| < \epsilon$, then $|f(x) - x^2| < \frac{1}{2}|x|^2$. By the triangle inequality this means that $f(x) > \frac{1}{2}|x|^2 \geq 0$ for $x \in (-\epsilon, \epsilon)$.

(e) (10 points) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

False. Let $a_n = \frac{1}{n}$, this series has $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$.

(f) (10 points) There is a sequence of continuous functions $f_n : [-1,1] \to \mathbb{R}$ converging uniformly to the function $f : [-1,1] \to \mathbb{R}$ given by $f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$.

False. For f to be the uniform limit of continuous functions, it must itself be continuous. The given f is not continuous as $\lim_{x\to 0} f(x)$ does not exist.

(g) (10 points) If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then for all bijections $m : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n=1}^{\infty} a_{m(n)}$ is convergent and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{m(n)}$.

False. If the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ convergent. However, there is a rearrangement $m : \mathbb{N} \to \mathbb{N}$ so that $\sum_{n=1}^{\infty}(-1)^{m(n)} \frac{1}{m(n)}$ diverges to infinity.

(h) (10 points) Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable. If $f(q) = 0$ for all rational numbers $q \in [0, 1]$, then $\int_0^1 f(x)dx = 0$.

True. Consider partitions $P_n = \left\{0 < \frac{1}{n} < \cdots < \frac{k}{n} < \cdots < 1\right\}$. Then $|P_n| \to 0$, and so $S(f, P_n, A) \to \int_0^1 f(x)dx$ for any choice of A, picking A to be the left endpoints, we see that $S(f, P_n, A) = 0$ and so $\int_0^1 f(x)dx = 0$.

2. (a) (10 points) State the intermediate value theorem.

Let $f : [a, b] \to \mathbb{R}$ be continuous with $f(a) < f(b)$, then for every $y \in [f(a), f(b)]$, there is an $x \in [a, b]$ so that $y = f(x)$.

(b) (15 points) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $I \subset \mathbb{R}$ is a compact interval, then $f(I)$ is compact interval.

Write $I = [a, b]$. Set $M = \sup_I f(x)$ and $m = \inf_I f(x)$. As I is compact and f continuous, there are $x, y \in I$ with $f(x) = m$ and $f(y) = M$. Hence, $M, m \in \mathbb{R}$. We have $\{m, M\} \subset f(I) \subset [m, M]$ and so if $m = M$ we have nothing to prove. If $m < M$, then either $x < y$ or $y < x$. WLOG we assume $x < y$. For any $z \in (m, M)$ there is a $z' \in (x, y) \subset I$ with $f(z') = z$. This implies that $f(I) \supset [m, M]$, that is $f(I) = [m, M]$.

3. (a) (10 points) State the mean value theorem.

Let $f : (a, b) \to \mathbb{R}$ be differentiable. For $a < x < y < b$, there is a $z \in (x, y)$ so that $f'(z) = \frac{f(y)-f(x)}{y-x}$.

(b) (15 points) Show that if $f : \mathbb{R} \to \mathbb{R}$ is *differentiable* and $f'(x) \geq x$, then $f(x) \leq f(0) + \frac{1}{2}x^2$ when $x \leq 0$.

Consider the function $g(x) = f(x) - f(0) - \frac{1}{2}x^2$. This function is also differentiable as $f(0) + \frac{1}{2}x^2$ is. One computes that $g'(x) = f'(x) - x > 0$, so g is non-decreasing. Moreover, $g(0) = 0$. Hence, $g(x) \leq 0$ for $x \leq 0$. Which proves the claim.

4. (a) (10 points) State one of the (equivalent) definitions of a function $f : [a, b] \to \mathbb{R}$ being Riemann integrable.

> f is Riemann integrable if it is bounded and for every $\epsilon > 0$, there is a $\delta > 0$, so that if P is a partition with $|P| < \delta$, then $Osc(f, P) = S^{+}(f, P) - S^{-}(f, P) < \epsilon$.

(b) (10 points) Give an example of a function $f : [0, 1] \to \mathbb{R}$ which is not Riemann integrable. You do not need to justify this.

> Consider Dirichlet's function $f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$

(c) (15 points) Using the definition from a), show that if $f : [a, b] \to \mathbb{R}$ is continuous, then it is Riemann integrable.

As f is continuous and $[a, b]$ is compact, f is uniformly continuous and is bounded. Using the uniform continuity of f, given an $\epsilon > 0$, pick $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. For any partition, $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ we have $S^+(f, P) = \sum_{i=1}^n M_i(x_i - x_{i-1})$ where $M_i = \sup_{[x_{i-1},x_i]f(x)}$ and $S^-(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1})$ where $m_i = \inf_{[x_{i-1},x_i]f(x)}$. By the continuity of f and compactness of $[x_{i-1}, x_i]$ we have $M_i = f(a_i)$ and $m_i = f(b_i)$. Hence, if $|P| < \delta$, then $Osc(f, P) = \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) < \epsilon$ as $|a_i - b_i| < \delta$.

5. Let $f : D \to \mathbb{R}$ be a function.

(a) (10 points) State the definition of f being (real) analytic.

f is real analytic, if D is open and for every $x_0 \in D$, there is a power series $\sum_{n=0}^{\infty} a_n(x_0)(x - x_0)^n$ with positive radius of converge and so that $f(x) = \sum_{n=0}^{\infty} a_n(x_0)(x - x_0)^n$ in a neighborhood of x_0.

(b) (10 points) Give an example of a function f that is infinitely differentiable (i.e. of class C^∞) but that is not real analytic. You do not need to justify your answer.

Let $f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$. This function is infinitely differentiable, but is not real analytic as f does not agree with any powerseries near $x = 0$.

(c) (15 points) Show that if D is an interval, f is real analytic and $f(x) = 0$ for all $x \in I$ for $I \subset D$ an open interval, then $f(x) = 0$ for all $x \in D$.

Hint: Consider the maximum interval containing I on which f vanishes. Using the Taylor polynomials at the endpoints prove this interval is D.

Set $D = (a, b)$ and $I = (c, d)$. Let $z_- = \inf\{x : f(z) = 0 \text{ for all } z \in (x, d)\}$ and set $z_+ = \sup\{x : f(z) = 0 \text{ for all } z \in (c, x)\}$. If $z_- = a$ and $z_+ = b$, then there is nothing to prove. Assume $z_- \neq a$ – that is, $z_- \in D$. There is a sequence $x_n \in (z_-, d)$ with $x_n \to z_-$. Notice, that as $f(x) = 0$ in all of (z_-, d) that $f^{(n)}(x_k) = 0$ for all n. As f is analytic, it is C^n for all n. Hence, passing to a limit and using the continuity of $f^{(n)}$ we see that $f^{(n)}(z_-) = 0$ – we use here that $z_- \in D$. As f is analytic and $z_- \in D$, there is an interval $R > 0$ so that $f(x) = \sum_{n=0}^{\infty} a_n (x - z_-)^n$ when $|x - z_-| < R$ and $x > a$. However, as $f^{(n)}(z_-) = 0$ for all n we have that $a_n = 0$ for all n. Hence, for any x with $|x - z_-| < R$ and $x > a_-$ we must have $f(x) = 0$. However, this contradicts our definition of z_- proving that $z_- = a$. That $z_+ = b$ is proved in the same way.