Solutions Midterm Exam 1 — Oct. 7, 2019

1. (a) (10 points) Give the formal definition of what it means for a set X to be countable infinite.

A set X is countable infinite if there exists a bijection $f : \mathbb{N} \to X$. Here $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers. The map f being a bijection means that, for every $x \in X$, there is a unique element $n \in \mathbb{N}$ so that $f(n) = x$.

(b) (10 points) Use mathematical induction to prove that, for every $n \in \mathbb{N}$ with $n \geq 4$, one has $2^n < n!$. Here $n! = 1 \cdot 2 \cdot \ldots \cdot n$ is the factorial of n.

In order to prove the inequality by induction, we must first establish it for the base case, for this problem that is $n = 4$. In this case one has $2^4 = 16 < 24 = 4!$. To complete the proof one supposes that one has already established $0 < 2^m < m!$ for a given $m \geq 4$. As $m \geq 4$, one has $m + 1 \geq 2 > 0$ and so, by the way order and multiplication of natural numbers interact, $2^{m+1} = 2^m \cdot 2 < m! \cdot (m + 1) = (m + 1)!$. Hence, by the principle of mathematical induction the inequality holds for every $m \geq 4$.

2. (15 points) Let $A \subset \mathbb{R}$ be a non-empty set with the property that if $x, y \in A$, then $x + y \in A$. Show that if A is bounded from below, then $\inf(A) \geq 0$.

Observe that as A is non-empty and bounded from below $x = \inf(A)$ is a real number. If $x < 0$, then, by the properties of an ordered field $x < \frac{x}{2} < 0$. Hence, as x is the infimum of A, $\frac{x}{2}$ is not a lower bound of A and so there is an element $y \in A$ so $y \leq \frac{x}{2} < 0$. Note that $2y = y + y \in A$ and so $w = 4y = 2y + 2y \in A$, by the hypotheses on A. However, this means $w = 4y \leq 2x < x$ by the ordered field properties. This contradicts the fact that x is a lower bound for A and so one concludes that $x = \inf(A) \geq 0$ as claimed.

3. (a) (10 points) State the formal definition of what it means for a sequence $\{x_n\}_{n=1}^{\infty}$ to converge to x.

A sequence $\{x_n\}_{n=1}^{\infty}$ converges to x if for every $\epsilon > 0$, there exists a $M \in \mathbb{N}$ so that if $n \geq M$, then $|x_n - x| < \epsilon$.

(b) (15 points) Show directly from the definition that if $x_n = \frac{n-2}{2n+4}$ for $n \in \mathbb{N}$, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $\frac{1}{2}$.

Given $\epsilon > 0$, pick M so $M\epsilon > 2$ which we can do by the Archimedean principle. For $n \geq M$ we have

$$\left|\frac{n-2}{2n+4} - \frac{1}{2}\right| = \left|\frac{n-2-(n+2)}{2n+4}\right| = \left|\frac{-4}{2n+4}\right| = 2\frac{1}{n+2}.$$

Hence, as $n + 2 > n \geq M > 0$ we have

$$\left|\frac{n-2}{2n+4} - \frac{1}{2}\right| \leq \frac{2}{M} < \epsilon.$$

As a consequence, $\lim_{n\to\infty} x_n = \frac{1}{2}$.

4. (a) (10 points) Give a bounded sequence $\{x_n\}_{n=1}^{\infty}$ so that $(\limsup_{n\to\infty} x_n)^2 \neq \limsup_{n\to\infty} x_n^2$.

Set $x_n = x_n = -1 + (-1)^n$ which satisfies $-2 \leq x_n \leq 0$. Clearly, $x_n^2 = 2 - 2(-1)^n$. One has $a_n = \sup\left\{x_k^2 : k \geq n\right\} = 4$ and $b_n = \sup\{x_k : k \geq n\} = 0$ and so it follows $\limsup_{n\to\infty} x_n^2 = \lim_{k\to\infty} a_k = 4$ while $(\limsup_{n\to\infty} x_n)^2 = (\lim_{k\to\infty} b_k)^2 = 0$.

(b) (15 points) Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. Show that if $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} x_n^2 = (\limsup_{n\to\infty} x_n)^2$.

Given a non-empty bounded set $A \subset [0, \infty)$ let $B = \left\{x^2 : x \in A\right\}$. One readily sees that B is also a bounded set. Moreover, for any $x \in A$ one has $0 \leq x \leq \sup A$. As $x \geq 0$, the properties of ordered fields imply that $x^2 \leq x(\sup(A)) \leq (\sup(A))^2$. Hence, $\sup(B) \leq (\sup(A))^2$.

In fact, one has $\sup(B) = (\sup(A))^2$. If $\sup(A) = 0$, then $A = \{0\}$ and $B = \{0\}$ and the equality is immediate. As such, one may assume $\sup(A) > 0$. In this case, as $0 < (1-\epsilon)\sup(A) < \sup(A)$ for every $\epsilon \in (0, 1)$ there is an element $y \in A$ so $y > (1-\epsilon)\sup(A) > 0$. By the ordered field axioms $y^2 > (1-\epsilon)^2(\sup(A))^2$. Hence, $\sup(B) \geq y^2 > (1-\epsilon)^2(\sup(A))^2$. As $\epsilon > 0$ can be taken as small as one likes one concludes that $\sup(B) \geq (\sup(A))^2$. Combined with the previous inequality this verifies the claim.

It follows that, as $x_n \geq 0$, $c_k = \sup\left\{x_n^2 : n \geq k\right\} = a_k^2$ where $a_k = \sup\{x_n : n \geq k\}$. Hence, by the limit laws and the fact that monotone sequences converge one has

$$\limsup_{n\to\infty} x_n^2 = \lim_{k\to\infty} c_k = \lim_{k\to\infty} a_k^2 = (\lim_{k\to\infty} a_k)^2 = (\limsup_{k\to\infty} x_k)^2.$$

Proving the claim.

5. (15 points) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined inductively by $x_1 = 2$ and $x_{n+1} = x_n - x_n^2 - 1$ for $n \geq 1$. Show this sequence cannot converge. (Hint: Suppose $\lim_{n\to\infty} x_n = x$, what equation would x satisfy?).

Suppose that the sequence does converge. Set $x = \lim_{n\to\infty} x_n$. By the limit laws this would imply that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(x_n - x_n^2 - 1\right) = \lim_{n\to\infty} x_n - \left(\lim_{n\to\infty} x_n\right)^2 - \lim_{n\to\infty} 1 = x - x^2 - 1.$$

It follows that x would have to satsify

$$x^2 = -1 < 0.$$

However, it follows from the properties of an ordered field that $x^2 \geq 0$. This is a contradiction and leads one to conclude that there is no such x. That is, the sequence can't converge.