Solutions Midterm Exam 1 — Feb. 23, 2015

1. Let $\sim$ be an equivalence relation on $\mathbb{N}$.

   (a) (10 points) Give the formal definition the equivalence class $[n]$ of an element $n \in \mathbb{N}$ with respect to $\sim$ and the formal definition of the quotient space $\mathbb{N}/\sim$ as a subset of the power-set $P(\mathbb{N})$.

   By definition, the equivalence class $[n] = \{m \in \mathbb{N} : n \sim m\} \subset \mathbb{N}$. The quotient space $X/\sim$ is just the set of all equivalence classes of $\sim$, that is $\mathbb{N}/\sim = \{[n] \in P(X) : n \in \mathbb{N}\}$.

   (b) (15 points) Show that if each equivalence class of $\sim$ contains only finitely many elements, then the quotient space $X = \mathbb{N}/\sim$ is countable.

   We first observe that each element $U \in X$ may be thought of as a subset $U \subset \mathbb{N}$. Furthermore, $\mathbb{N} = \cup_{U \in X} U$. In particular, X cannot be finite as if it were, $\mathbb{N}$ would be the finite union of finite sets. We now construct a map

   $$\phi : X \to \mathbb{N}$$

   as follows for each $U \in X$ let $\phi(U) = \min\{x : x \in U\}$. This map is well-defined by the well-ordering principle. If $\phi(U) = \phi(U') = n$, then $n \in U \cap U'$ and so $U = U'$. That is, ϕ is an injection. Let $\phi(X) = \{n \in \mathbb{N} : n = \phi(U) \text{ for some } U \in X\}$. Clearly, $\phi : X \to \phi(X)$ is a bijection and so $\phi(X)$ is not finite. As $\phi(U) \subset \mathbb{N}$, one must have that $\phi(U)$ is countable and hence so is X.

2. (a) (10 points) State the formal definition of Cauchy sequence (of real numbers).

A sequence $\{a_n\}$ of real numbers is Cauchy if and only if for every $\epsilon > 0$, there is an $m \in \mathbb{N}$ so that if $n, k > m$, then $|a_n - a_k| < \epsilon$.

(b) (15 points) Show directly from the definition, that if $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is bounded in the sense that there is an $N > 0$ so that $|a_n| < N$ for all $n \in \mathbb{N}$.

By definition, there is an m, so that for all $n, k > m$ one has $|a_n - a_k| < 1$. Let $N = 1 + \max\{|a_1|, |a_2|, \ldots, |a_{m+1}|\}$. Clearly, for $i \leq m$, $|a_i| < N$. For $i > m$, we have $|a_i - a_{m+1}| < 1$. The triangle inequality gives that $|a_i| - |a_{m+1}| \leq |a_i - a_{m+1}|$. Hence, $|a_i| < 1 + |a_{m+1}| \leq N$.

3. (a) (10 points) Give an example of a set $X \subset \mathbb{R}$ which is bounded from above and does not contain its least upper bound.

> The set $[0, 1)$ is bounded from above by 1. This is also its least upper bound as any upper bound must be greater than $1 - \epsilon$ for all $\epsilon > 0$.

(b) (15 points) Let $A, B \subset \mathbb{R}$ be non-empty sets which are both bounded from above. Define $A+B = \{z \in \mathbb{R} : z = a + b \text{ for some } a \in A \text{ and some } b \in B\}$. Show that $\sup(A + B) = \sup A + \sup B$.

> Observe that if u is an upper bound for A and v is an upper bound for B, then for all $a \in A$ and $b \in B$ we have $a \leq u$, $b \leq v$ and so $a+b \leq u+v$. Hence, $u+v$ is an upper bound for $A+B$. This means that $\sup A+\sup B$ is an upper bound for $A+B$ and so $\sup(A+B) \leq \sup A+\sup B$. For all $\epsilon > 0$, the definition of sup ensures there is an $a \in A$ and a $b \in B$ so that $a > \sup A - \epsilon$ and $b > \sup B - \epsilon$. In particular, there is a $z \in A + B$ so that $z > \sup A + \sup B - 2\epsilon$ (indeed, take $z = a + b$). Hence, $\sup(A + B) \geq \sup A + \sup B - 2\epsilon$ for all $\epsilon > 0$. This implies that $\sup(A + B) \geq \sup A + \sup B$ and the claim follows.

4. (a) (10 points) Give an example of bounded sequences $\{x_n\}$ and $\{y_n\}$ so that x is a limit point of $\{x_n\}$ and y is a limit point of $\{y_n\}$, but $x + y$ is *not* a limit point of the sequence $\{x_n + y_n\}$.

Let $x_n = (-1)^n$ and $y_n = -(-1)^n$. Then $\{x_n\}$ and $\{y_n\}$ have both ± 1 as a limit point. However, $x_n + z_n = 0$ for all n so 0 is the only limit point of $\{x_n + z\}$ is zero. In particular, $2 = 1 + 1$ is not a limit point.

(b) (15 points) Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Show that if z is a limit point of $\{x_n + y_n\}$, then there is a limit point x of $\{x_n\}$ and a limit point y of $\{y_n\}$ so that $z = x + y$. (Hint: Use the fact that limit points are limits of subsequences and that subsequences of convergent sequences converge).

As z is a limit point of there is a subsequence $z'_n = z_{\phi(n)}$ so that $\lim_{n\to\infty} z'_n = z$. Let $x'_n = x_{\phi(n)}$ and let $x' = \limsup_{n\to\infty} x'_n$. As x' is a limit point of $\{x'_n\}$, there is a subsequence $x''_n = x'_{\psi(n)}$ so that $\lim_{n\to\infty} x''_n = x'$. Observe that $x''_n = x_{\phi(\psi(n))}$ is a subsequence and so $\{x''_n\}$ is a subsequence of $\{x_n\}$ – in particular, x' is a limit point of $\{x_n\}$. Now let $y''_n = y'_{\psi(n)}$ so $\{y''_n\}$ is a subsequence of $\{y'_n\}$ (and hence also of $\{y_n\}$). Let $y'' = \limsup_{n\to\infty} y''_n$. As y'' is a limit point of $\{y''_n\}$, there is a subsequence $y'''_n = y''_\varphi$ so $y'' = \lim_{n\to\infty} y'''_n = y''$. Again, y'' is a limit point of $\{y_n\}$. Let $x'''_n = x''_{\varphi(n)}$ and $z'''_n = x'''_n + y'''_n$. Observe, $\{x'''_n\}$ is a subsequence of $\{x''_n\}$ and $\{z'''_n\}$ is a subsequence of $\{z'_n\}$. Hence, $\lim_{n\to\infty} x'''_n = \lim_{n\to\infty} x''_n = x'$, and $\lim_{n\to\infty} z'''_n = \lim_{n\to\infty} z'_n = z$ and so $z = \lim_{n\to\infty} z'''_n = \lim_{n\to\infty} x'''_n + \lim_{n\to\infty} y'''_n = x' + y''$.