Solutions Midterm Exam 2 — Nov. 11, 2019

1. (a) (10 points) State the intermediate value theorem.

The intermediate value theorem states that if $f : [a,b] \to \mathbb{R}$ is continuous and $f(a) < y < f(b)$ or $f(b) < y < f(a)$, then there is a $c \in (a,b)$ so $f(c) = y$.

(b) (10 points) Show that if $f : (-2,2) \to \mathbb{R}$ is differentiable and $f(-1) = f(1) = 0$ and $f'(-1) = f'(1) = 1$, then f must have a zero in $(-1,1)$.

As $f'(-1) = 1$, this means that there is a $\delta > 0$ so if $0 < |x-(-1)| < \delta$ one has $\left|\frac{f(x)-f(-1)}{x-(-1)} - 1\right| < \frac{1}{2}$. This means that for $x \in (-1,-1+\delta)$ one has $f(x) > \frac{1}{2}(x+1) > 0$. In particular, we have a $a \in (-1,0)$ so $f(a) > 0$. A similar argument gives a $b \in (0,1)$ so $f(b) < 0$. As f is differentiable on $(-2,2)$ it is continuous on $(-2,2)$ and hence also on $[a,b]$. Hence, we can apply the intermediate value theorem to f to see that there is a $c \in (a,b) \subset (-1,1)$ so $f(c) = 0$.

2. (a) (10 points) State the definition of uniform continuity on an interval I for a function $f : I \to \mathbb{R}$.

The function f is uniformly continuous on I if, for every $\epsilon > 0$, there is a $\delta > 0$ so that if $x, y \in I$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

(b) (10 points) Give an example of a continuous function $f : (0, 1) \to \mathbb{R}$ that is not uniformly continuous. Please justify your answer.

The function $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$ as it is the quotient of two continuous functions that don't vanish. It is not uniformly continuous as if $x, y \in (0, 1)$ satisfy

$$|f(x) - f(y)| < 1,$$

then

$$|x - y| < xy.$$

In particular, there can be no $\delta > 0$ so $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$. Indeed, if there was such a $\delta > 0$ then one could pick $x = \min\{\delta/2, 1/2\}$ and $y = x/2 \leq \frac{1}{4}$. This gives $|x - y| = \frac{x}{2}$ and so $\frac{x}{2} < xy \leq \frac{x}{4}$, which contradicts, $x > 0$.

(c) (10 points) Show that if $f : (a, b) \to \mathbb{R}$ is uniformly continuous, then f is bounded.

As f is uniformly continuous, there is a $\delta > 0$ so that if $x, y \in (a, b)$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < 1$. Pick $a' \in (a, a + \delta)$ and $b' \in (b - \delta, b)$ so $a' < b'$. Observe that f is continuous on $[a', b']$ and so, by the min-max theorem, there is a constant M so $|f(x)| \leq M$ for $x \in [a', b']$. Observe, that if $x \in (a, a')$, then one has $|x - a'| < \delta$ and so

$$|f(x)| = |f(x) - f(a') + f(a')| \leq |f(x) - f(a')| + |f(a')| \leq 1 + M.$$

In a similar fashion if $x \in (b', b)$, then

$$|f(x)| \leq 1 + M.$$

It follows that for every $x \in (a, b)$ one has $|f(x)| \leq 1 + M$.

3. (a) (10 points) State the mean value theorem.

The mean value theorem states that if $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b), then there is a $c \in (a, b)$, so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) (10 points) Show that if $f : (a, b) \to \mathbb{R}$ is differentiable and satisfies $|f'(x)| \leq C$ for all $x \in (a, b)$, then $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in (a, b)$.

If $x = y$, then there is nothing to show. With out loss of generality we may assume $x < y$ (as the argument is identical if $y < x$). As f is differentiable on (a, b) it is also continuous. By the mean value theorem applied to f on $[x, y]$, one has $f(y) - f(x) = f'(c)(y - x)$ for some $c \in (x, y)$. Hence,

$$|f(y) - f(x)| = |f'(c)||y - x| \leq C|y - x|.$$

the desired result immediately follows from this bound.

4. (a) (10 points) State the definition of the upper and lower Darboux integrals, $\overline{\int_a^b} f(x)dx$ and $\underline{\int_a^b} f(x)dx$ of a bounded function $f : [a,b] \to \mathbb{R}$.

The upper Darboux integral is given by

$$\overline{\int_a^b} f(x)dx = \inf \{U(P,f) : P \text{ partition of } [a,b]\}$$

where

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\}$$

is a partition of $[a,b]$ and

$$U(P,f) = \sum_{i=1}^{N} M_i \Delta x_i$$

is an upper Darboux sum. Here $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$ and $\Delta x_i = x_i - x_{i-1}$.
Likewise, a lower Darboux integral is given by

$$\underline{\int_a^b} f(x)dx = \sup \{L(P,f) : P \text{ partition of } [a,b]\}$$

where

$$L(P,f) = \sum_{i=1}^{N} m_i \Delta x_i$$

is the lower Darboux sum. Here $m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$.

(b) (5 points) Given an example of a bounded function $f : [0,1] \to \mathbb{R}$ so $\overline{\int_0^1} f(x)dx \neq \underline{\int_0^1} f(x)dx$. That is, give an example of a bounded function on $[0,1]$ that is not Riemann integrable. You do not have to justify your answer.

An example is given by the function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \in [0,1] \backslash \mathbb{Q} \end{cases}$$

Indeed, in this case $U(P,f) = 1$ for every partition and so the upper Darboux integral is 1, while $L(P,f) = 0$ for every partition and so the lower Darboux integral is 0.

(c) (15 points) Show that if $f : [a, b] \to \mathbb{R}$ is monotone decreasing, then $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$, that is, f is Riemann integrable.

First observe that as f is monotone decreasing one has $f(b) \leq f(x) \leq f(a)$ for all $x \in [a, b]$ and so f is bounded. Let $M > 0$ be a bound on f so $|f(x)| \leq M$ for $x \in [a, b]$. As a consequence, the upper and lower Darboux integrals both exist. As we showed in class, for any partition, P one has

$$0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq U(P, f) - L(P, f).$$

It follows that for any $\epsilon > 0$ we just need to find a partition P_ϵ fo $[a, b]$ so $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$. To that end, given an $\epsilon > 0$, fix a partition $P_\epsilon = \{a = x_0 < x_1 < \ldots < x_n = b\}$ so that $\Delta x_i = \frac{b-a}{n} < \epsilon/2M$ for $1 \leq i \leq n$.

Observe that as f is monotone decreasing one has

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = f(x_{i-1})$$

while

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = f(x_i)$$

We compute that

$$U(P_\epsilon, f) - L(P_\epsilon, f) = \sum_{i=1}^n (M_i - m_i)\Delta x_i = \sum_{i=1}^n (f(x_{i-1}) - f(x_i)) \frac{b-a}{n} = \frac{b-a}{n} \sum_{i=1}^n (f(x_{i-1}) - f(x_i))$$

As this is a telescoping sum, this yields

$$U(P_\epsilon, f) - L(P_\epsilon, f) = \frac{b-a}{n}(f(a) - f(b)) < (f(a) - f(b))\epsilon/2M \leq \epsilon.$$

Hence, sending $\epsilon \to 0$ one sees that the upper and lower Darboux integrals agree as claimed.