Solutions Midterm Exam 2 — Apr. 6, 2015

1. (a) (10 points) State the intermediate value theorem for continuous functions.

   A continuous function f on a closed interval $[a, b]$ assumes all values between $f(a)$ and $f(b)$.

   (b) (15 points) Let $f : [0, 1] \to [0, 1]$ be continuous. Show that there is at least one $x \in [0, 1]$ so that $f(f(x)) = x$.

   Consider function $g(x) = f(x) - x$ this function is continuous as f is as well. If $g(0) = f(0) = 0$ or $g(1) = f(1) - 1 = 0$ then $f(f(0)) = 0$ or $f(f(1)) = 1$ respectively. Therefore it remains to study the case where $g(0) > 0$ and $g(1) < 0$, then by IVT, there exists $x \in (0, 1)$, $g(x) = 0$, so $f(x) = x$ which implies $f(f(x)) = x$.

2. (a) (10 points) Give the formal definition for a function $f : D \to \mathbb{R}$ to be uniformly continuous.

We say f is uniformly continuous on D if for every m there exists n such that for any $x, y \in D$ with $|x - y| < 1/n$, we have $|f(x) - f(y)| < 1/m$.

(b) (15 points) Using the formal definition directly, show that $f(x) = \frac{1}{x}$ is uniformly continuous as a map $f : (1, 2) \to \mathbb{R}$.

Consider any error $1/m$, and two points $x, y \in (1, 2)$, then by the fact $|xy| > 1$

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| < |y - x|$$

, thus if we take $1/n = 1/m$, then for any $x, y \in (1, 2)$ with $|x - y| < 1/n$, we would have

$$\left|\frac{1}{x} - \frac{1}{y}\right| < |y - x| < \frac{1}{m}.$$

So f is uniformly continuous on $(1, 2)$.

3. (a) (10 points) State the mean value theorem.

If f is continuous on $[a, b]$ and differentiable on (a, b), then $f'(x_0) = (f(b) - f(a))/(b - a)$ for some x_0 in (a, b).

(b) (15 points) Suppose that $f : (-1, 1) \to \mathbb{R}$ is differentiable and that $f(0) = 0$ and $|f'(x)| \leq |x|^3$. Show that $|f(x)| \leq x^4$.

By the mean value theorem, there exists some x_0 with $0 \leq |x_0| \leq |x|$, such that

$$|f(x)| = |f(x) - f(0)| \leq |f'(x_0)||x - 0| \leq |x_0|^3|x| \leq x^4.$$

4. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and C^2 on $(0, 1)$ and $f(0) = f(1) = 0$.

   (a) (10 points) Give an example of such an f that has a strict local maximum at some point $x_0 \in (0, 1)$ but which has $f''(x_0) = 0$.

   Basically we want to modify $-x^4$ as our f, since it has local maximum at 0 while its second order derivative at 0 is 0.
   So take $f(x) = -(x - 1/2)^4 + 1/16$.

   (b) (15 points) Show that if $f''(x) \geq 0$ for $x \in (0, 1)$, then $f(x) \leq 0$ for $x \in (0, 1)$. (Hint: Consider $g_\epsilon(x) = f(x) + \epsilon x(x - 1)$ for $\epsilon > 0$ and consider what happens as $\epsilon \to 0$).

   Clearly for any $\epsilon > 0$, $g''_\epsilon(x) = f''(x) + 2\epsilon > 0$, together with the fact $g_\epsilon(0) = g_\epsilon(1) = 0$, we know that the graph of $g(x)$ lies below the secant line connecting (0,0) and (1,0), which tells us $g_\epsilon(x) < 0$ for any $x \in (0, 1)$. Taking $\epsilon \to 0$ gives that $f(x) \leq 0$ for any $x \in (0, 1)$.