Practice Midterm 2

1. (10 pts each) True or false; justify as much as you can.

a. If $f(x)$, $g(x)$ are continuous functions on [0,1] which agree at every rational, then $f = g$ on [0,1].

b. If $|f(x)|$ is continuous at x_0 then $f(x)$ is continuous at x_0.

c. If f is a strictly monotone function on [0,1 with range an interval, then f is one to one.

d. Let f be continuous on $\mathbb{R}$. Then the inverse image of an open interval is an open interval.

e. If $f(x)$ is uniformly continuous on $\mathbb{R}$ and $\{x_n\}$ is a Cauchy sequence, then so is $\{f(x_n)\}$.

f. There exists a continuous bijection map $f : [0, 1) \to \mathbb{R}$.

2. Let $f : [0, 1] \to [0, 1]$ be continuous. Show that the equation $f(x) = x$ has at least one solution in [0,1].

3. Let $f(x)$ be a C^1 function on $\mathbb{R}^+$ and satisfy $f'(x) > f(x)$, $f(0) = 0$. Show that $f(x) > 0$ for $x > 0$.

4. Let $f(x)$ be strictly increasing and continuous on $[0, \infty)$ with f(0)=0. Show that

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab \ .$$

When does equality hold? Hint: Draw a picture and interpret geometrically.

5. Let $f(x)$ be C^3 on an interval I . Suppose $a_0 < a_1 < a_2$ are points of I and $f(a_0) = f(a_1) = f(a_2) = f'(a_2) = 0$. Show there is a point $c \in I$ where $f'''(c) = 0$.

6. Let $f(x)$ be continuous on $[0, \infty)$ and assume that $L = \lim_{x\to+\infty} f(x)$ exists and is finite. Show that f is bounded. (Recall $L = \lim_{x\to+\infty} f(x)$ means that give $\varepsilon > 0$, $\exists N = N(\varepsilon)$ such that $x > N$ implies $|f(x) - L| < \varepsilon$.)

7. Let $f(x)$ be Riemann integrable on [0,1] and assume that $f(x) = 0$ when x is rational. Show that $\int_0^1 f(x)dx = 0$. Note that $f(x)$ is assumed bounded but nothing is assumed about the values of $f(x)$ when x is irrational.

8. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n},\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$

Show that $f(x)$ is Riemann integrable.