1. Suppose that \( f : D^*_1(0) \to \mathbb{C} \) is a holomorphic function satisfying \( |f(z)| \leq C|z|^{-3/2} \). Here \( D^*_1(z) = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) and \( C > 1 \).

(a) (15 points) Show that \( f \) has either a simple pole at \( z = 0 \) or a removable singularity. Hint: Consider \( h(z) = z^2 f(z) \).

Let \( h(z) = z^2 f(z) \). We have \( |h(z)| = |z^2| |f(z)| \leq C|z|^{1/2} \leq C \) on \( D^*_1 \). This means \( h \) is bounded and holomorphic in \( D^*_1 \). Hence, by the Riemann removable singularities theorem, \( 0 \) is a removable singularity for \( h \) and so \( h \) extends to a holomorphic function \( H(z) \) on all of \( D_1 \). Notice, that \( |H(0)| = \lim_{z \to 0} |H(z)| \leq \lim_{z \to 0} C|z|^{1/2} = 0 \) and so \( H \) has a zero at \( z = 0 \). In particular, we can write \( H(z) = zG(z) \) where \( G \) is holomorphic in \( D_1 \). Clearly, \( f(z) = z^{-1}G(z) \) and so either \( f \) has a simple pole at \( z = 0 \) (if \( G(0) \neq 0 \)) or \( f \) has a removable singularity at \( z = 0 \) (if \( G(0) = 0 \)).

(b) (5 points) Show that if, in addition, \( |f(z)| \geq |z|^{-1/2} \), then \( f \) has a simple pole at \( z = 0 \).

The lower bound ensures that \( \lim_{z \to 0} |f(z)| \geq \lim_{z \to 0} |z|^{-1/2} = \infty \) and so \( f \) has a pole at \( z = 0 \). By the previous part this pole is a simple pole.
2. Compute the following contour integrals.

(a) (10 points) \( \int_{\partial D_1(0)} \frac{e^{z^2} \cos(z)}{z^2} \, dz \)

As the integrand is meromorphic in \( D_1 \), the residue theorem gives
\[
\int_{\partial D_1(0)} \frac{e^{z^2} \cos(z)}{z^2} \, dz = 2\pi i \text{Res}_{z=0} e^{z^2} \cos(z) = 0.
\]

In order to get the last equality we note that, \( \frac{e^{z^2} \cos(z)}{z^2} \) has a pole of order 2 at \( z = 0 \). Hence, in order to compute the residue we use the first few terms in the Taylor expansion
\[
e^{z^2} = 1 + z^2 + O(z^4)
\]
and
\[
\cos(z) = 1 + \frac{z^2}{2} + O(z^4)
\]
to see that
\[
\frac{e^{z^2} \cos(z)}{z^2} = \frac{1}{z^2} + \frac{3}{2} + O(z^2)
\]
and so the residue at \( z = 0 \) is zero.

One could also use the residue formula for higher order poles
\[
\text{Res}_{z=0} \frac{e^{z^2} \cos(z)}{z^2} = \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz} \left( \frac{e^{z^2} \cos(z)}{z^2} \right) = \lim_{z \to 0} \left( 2ze^{z^2} - \sin(z) \right) = 0.
\]

(b) (10 points) Let \( f \) be holomorphic on \( D_3(0) \) and suppose \( f(1) = f'(1) = -1, \int_{\partial D_2(0)} \frac{f(\zeta)}{(\zeta - 1)^2} \, d\zeta \).

As \( f \) is holomorphic in \( D_3 \), the generalized Cauchy integral formula gives
\[
-1 = f'(1) = \frac{1}{2\pi i} \int_{\partial D_2(0)} \frac{f(\zeta)}{(\zeta - 1)^2} \, d\zeta.
\]
Hence,
\[
\int_{\partial D_2(0)} \frac{f(\zeta)}{(\zeta - 1)^2} \, d\zeta = -2\pi i.
\]
One could also use the residue theorem.
(c) (10 points) Let \( f \) be holomorphic on \( D_2(0) \) and \( f(1) = 2 \) and \( f(-1) = 1 \). Compute \( \int_{\partial D_2(0)} \frac{f(z)}{z^2-1} \, dz \).

We note that \( z^2 - 1 = (z - 1)(z + 1) \) and so \( \frac{f(z)}{z^2-1} \) has simple poles at \( z = 1 \) and \( z = -1 \). Hence, the residue theorem implies that

\[
\int_{\partial D_2(0)} \frac{f(z)}{z^2-1} \, dz = 2\pi i \left( \text{Res}_{z=1} \frac{f(z)}{z^2-1} + \text{Res}_{z=-1} \frac{f(z)}{z^2-1} \right).
\]

As the poles are simple we have

\[
\text{Res}_{z=1} \frac{f(z)}{z^2-1} = \lim_{z \to 1} (z - 1) \frac{f(z)}{z^2 - 1} = \lim_{z \to 1} \frac{f(z)}{z + 1} = \frac{f(1)}{1+1} = 1
\]

and

\[
\text{Res}_{z=-1} \frac{f(z)}{z^2-1} = \lim_{z \to -1} (z + 1) \frac{f(z)}{z^2 - 1} = \lim_{z \to -1} \frac{f(z)}{z - 1} = \frac{f(-1)}{-1-1} = -\frac{1}{2}
\]

Hence,

\[
\int_{\partial D_2(0)} \frac{f(z)}{z^2-1} \, dz = \pi i.
\]
3. Explain why there is no holomorphic function with the given domain and properties.

   (a) (10 points) A \( f : \mathbb{C} \to \mathbb{C} \) with \( f(5) = 0 \) and \( f\left(\frac{1}{n}\right) = 1 \) for all \( n \in \mathbb{Z}, \; n \geq 1 \). Hint: what is happening at \( z = 0 \).

We observe that \( \lim_{n \to \infty} \frac{1}{n} = 0 \) and hence 0 is a point of accumulation of the sequence \( \{\frac{1}{n}\}_{n \geq 1} \). This means that \( f \) must identically be equal to 1 by the analytic continuation property of holomorphic functions. This is inconsistent with \( f(5) = 0 \) and so there can be no such holomorphic \( f \).

(b) (10 points) A \( f : D_2(0) \to \mathbb{C} \) with \( f(0) = -2 \) and \( |f(z)| \leq 1 \) on \( \partial D_1(0) \)

As \( f \) is holomorphic on \( D_2 \) it is continuous on \( \bar{D}_1 \). Hence, by the maximum modulus principle one should have

\[
2 = |f(0)| \leq \max_{\partial D_1} |f(z)| \leq 1.
\]

As this is absurd, there can be no such holomorphic \( f \). One could also see this using the Cauchy integral formula.
(c) (10 points) A \( f : D_1^*(0) \to \mathbb{C} \) with \( 1 \leq |f(z)| \) for all \( z \in D_1^*(0) \) and so that \( \lim_{z \to 0} |f(z)| \) does not exist. Here \( D_1^*(0) = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \).

By hypotheses, \( f(z) \) never vanishes on \( D_1^* \) and so \( g(z) = \frac{1}{f(z)} \) is holomorphic on \( D_1^* \). Notice that

\[
|g(z)| = \frac{1}{|f(z)|} \leq 1
\]

Hence, \( g \) is bounded and so, by the Riemann removable singularity theorem, \( g \) extends to a holomorphic function of \( D_1 \). As such \( \lim_{z \to 0} |g(z)| = |g(0)| \) exists. This means \( \lim_{z \to 0} |f(z)| \) also exists (though could be \( \infty \) if \( g(0) = 0 \)) and so there is no such \( f \).

Alternatively, one could use that \( \lim_{z \to 0} |f(z)| \) does not exist only when \( f \) has an essential singularity at \( z = 0 \). The hypotheses that \( 1 \leq |f(z)| \) contradicts the Casorati-Weierstrass theorem and so shows that there can be no such \( f \).
4. Show (by construction) that there is a holomorphic function $f$ with the given properties.

(a) (10 points) A simply connected domain $\Omega$ and an $f : \Omega \to \mathbb{C}$ so that for all $z \in \Omega$, $(f(z))^2 = z$ and $f(1) = 1$ while $f(4) = -2$. Hint: Draw the right domain and use the corresponding logarithm.

Pick a simply connected domain $\Omega$ with the property that $1 \in \Omega$, $4 \in \Omega$, $0 \not\in \Omega$ and $\Omega$ “winds” clockwise one around 0. (e.g., consider a small neighborhood of the curve $t \mapsto (t + 1)^2 e^{2\pi i t}$, $t \in [0, 1]$). For such $\Omega$, $\log_\Omega(1) = 0$, while $\log_\Omega(4) = \log 4 + 2\pi i = 2 \log 2 + 2\pi i$.

Letting $f(z) = e^{\frac{1}{2} \log_\Omega(z)}$ one has that $f$ is holomorphic on $\Omega$ and

$$(f(z))^2 = \left(e^{\frac{1}{2} \log_\Omega(z)}\right)^2 = e^{\log_\Omega(z)} = z.$$ 

Moreover, $f(1) = e^{\frac{1}{2} \log_\Omega(1)} = e^0 = 1$ while $f(4) = e^{\frac{1}{2} \log_\Omega(4)} = e^{\log 2 + \pi i} = -2$. Verifying the claim.

(b) (10 points) A $f : \mathbb{C} \to \mathbb{C}$ with simple zeros at $z = 2^n$ for all $n \in \mathbb{Z}$, $n \geq 0$.

Consider the infinite product

$$f(z) = \prod_{n=0}^\infty \left(1 - \frac{z}{2^n}\right).$$

We note that this product converges uniformly on any $D_R$ as on this disk

$$\sum_{n=0}^\infty \frac{|z|}{2^n} = |z| \sum_{n=0}^\infty \frac{1}{2^n} \leq R \sum_{n=0}^\infty \frac{1}{2^n} < \infty.$$ 

Moreover, as this sum is finite this product only has simple zeros at $z = 2^n$ when $n = 0, 1, \ldots, \infty$. As such the product gives an entire function with simple zeros only at $z = 2^n$.

Alternatively, one can observe that a function that has simple zeros at (say) every integer numbers also has simple zeros at each $2^n$. For instance, $f(z) = \sin(\pi z)$ is an example.