1. (15 points) Let \( f : [0, 4] \to \mathbb{R} \) increase from \(-1\) to \(0\) on \([0, 1]\) and decrease from \(1\) to \(-1\) on \((1, 4]\) (i.e., \(f(0) = -1, f(1) = 0, \lim_{x \to 1^+} f(x) = 1, f(4) = -1\) and \(f\) is monotone on \([0, 1]\) and \((1, 4]\)). Show that \(f\) is a BV function and compute its total variation \(V_0^4 f\).

First observe that, as \(\lim_{x \to 1^+} f(x) = 1\), for any \(\epsilon > 0\) there is a \(\delta > 0\) so if \(x \in (1, 1 + \delta)\), then \(f(x) \geq 1 - \epsilon > 0\). Now let \(P = \{x_0, \ldots, x_n\}\) be any partition of \([0, 4]\). Up to taking a a finer partition, \(P_\epsilon\) one may assume that there are is an index \(n > i_1 > 0\) so \(x_{i_1} = 1\) and so \(1 \geq f(x_{i_1+1}) \geq 1 - \epsilon > 0\). Notice that \(f\) is increasing on \([x_0, x_{i_1}]\) and is decreasing on \((x_{i_1}, 1]\).

Hence,

\[ V(f, P) \leq V(f, P_\epsilon) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \]

\[ = \sum_{i=1}^{i_1} (f(x_i) - f(x_{i-1})) + |f(x_{i_1+1}) - f(x_{i_1})| - \sum_{i=i_1+2}^{n} (f(x_i) - f(x_{i-1})) \]

As two of these sums are telescoping one has

\[ V(f, P_\epsilon) = f(x_{i_1}) - f(x_0) + f(x_{i_1} + 1) - f(x_n) + f(x_{i_1} + 1) - f(x_{i_1}) \]

As \(f(x_{i_1}) = 0\) and \(1 \geq f(x_{i_1+1}) \geq 1 - \epsilon\) one has

\[ V(f, P_\epsilon) = f(x_{i_1}) - f(x_0) + f(x_{i_1} + 1) - f(x_n) + f(x_{i_1} + 1) - f(x_{i_1}) \]

and hence

\[ 4 - 2\epsilon \leq V(f, P_\epsilon) \leq 4. \]

This implies that

\[ V_0^4 f = \sup \{V(f, P) : P\} = 4 \]

and so \(f\) is BV.
2. (a) (10 points) Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ that is not Riemann integrable on $[0, 1]$, i.e. so $f \notin \mathcal{R}[0, 1]$. Remember to justify your answer.

The function

$$f(x) = \begin{cases} 1 & x \in [0, 1] \cap \mathbb{Q} \\ 0 & x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

is bounded, but is not Riemann integrable. This is because for any partition $P$ of $[0, 1]$ one has $U(f, P) = 1$ while $L(f, P) = 0$.

(b) (10 points) Let $\alpha : [-1, 1] \to \mathbb{R}$ be given by

$$\alpha(x) = \begin{cases} -1 & x \in [-1, 0] \\ 1 & x \in (0, 1]. \end{cases}$$

Give an example of a bounded function $f : [-1, 1] \to \mathbb{R}$ with a finite number of discontinuities that is not Riemann-Stieltjes integrable with respect to $\alpha$, i.e., so $f \notin \mathcal{R}_\alpha[-1, 1]$. Remember to justify your answer.

Let $f(x) = \alpha(x)$. Clearly $f$ has exactly one discontinuity at $x = 0$, but $f \notin \mathcal{R}_\alpha[-1, 1]$. The reason for this is if $P$ is a partition of $[-1, 1]$ which contains 0 one has $U_\alpha(f, P) = 2$ while $L_\alpha(f, P) = -2$. 
(c) (15 points) Show that if \( f : [-1, 1] \to \mathbb{R} \) is continuous, then is Riemann-Stieltjes integrable with respect to the \( \alpha \) from part b), i.e., \( f \in \mathcal{R}_\alpha[-1, 1] \). Compute \( \int_{-1}^{1} f \, d\alpha \).

Given \( \epsilon > 0 \) pick \( \delta > 0 \) so for \( |x| \leq \delta \), \( |f(x) - f(0)| \leq \frac{\epsilon}{2} \). Pick any partition \( P \) of \([-1, 1]\). Up to passing to a finer partition, \( P_\epsilon \), one may assume that \( 0 = x_{i_1} \in P_\epsilon \) and and \( x_{i_1} < x_{i_1+1} < \delta \). Notice that for \( i = i_1 \) one has \( \alpha(x_{i+1}) - \alpha(x_i) = 2 \), while for \( i \neq i_1 \), \( \alpha(x_{i+1}) - \alpha(x_i) = 0 \).

On \([x_{i_1}, x_{i_1+1}]\) one has

\[
f(0) - \epsilon/2 \leq \inf_{[x_{i_1}, x_{i_1+1}]} f(x) \leq \sup_{[x_{i_1}, x_{i_1+1}]} f(x) \leq f(0) + \epsilon/2.
\]

Hence,

\[
2f(0) + \epsilon \geq U_\alpha(f, P_\epsilon) \geq L_\alpha(f, P_\epsilon) \geq 2f(0) - \epsilon.
\]

As \( \epsilon > 0 \) is arbitrary, and \( \alpha \) is increasing this means that

\[
2f(0) \geq \inf_P U_\alpha(f, P) \geq \sup_P L_\alpha(f, P) \geq 2f(0).
\]

And so one has equality throughout. This implies \( f \in \mathcal{R}_\alpha[-1, 1] \) and

\[
\int_{-1}^{1} f(t) \, dt = 2f(0).
\]
3. (a) (10 points) Show by example that it is not true that if $\alpha$ is bounded on $[a,b]$ and $f$ is continuous on $[a,b]$, then $\int_a^b |f|d\alpha = 0$ implies $f$ identically vanishes. Remember to justify your answer.

Let $\alpha$ be the weight from problem 2 b). Let $f(x) = x^2$ so $f$ is continuous and $|f| = f$. By the computation of problem 2 c) one has

$$\int_{-1}^1 |f|d\alpha = |f(0)| = 0$$

but clearly $f$ does not identically vanish.
(b) (15 points) Show that if \( \alpha : [a, b] \to \mathbb{R} \) is strictly increasing (i.e., \( x > y \Rightarrow \alpha(x) > \alpha(y) \)), and \( f : [a, b] \to \mathbb{R} \) is continuous, then \( \int_a^b |f| d\alpha = 0 \) implies \( f \) identically vanishes.

Suppose \( f(x_0) \neq 0 \) at some point \( x_0 \in [a, b] \). By the continuity of \( f \) we may suppose \( x_0 \in (a, b) \) and up to replacing \( f \) by \(-f\) we may assume \( f(x_0) > 0 \). The continuity of \( f \) implies that there is a \( \delta > 0 \) so \( I = [x_0 - \delta, x_0 + \delta] \subset [a, b] \) and \( f(x) \geq \frac{1}{2} f(x_0) > 0 \) on \( I \). As \( \alpha \) is increasing and \( |f(t)| \geq 0 \) we have, for any \( a \leq x < y \leq b \) that

\[
\int_x^y |f| d\alpha \geq 0.
\]

Using properties of the integral, we have

\[
\int_a^b |f| d\alpha = \int_a^{x_0 - \delta} |f| d\alpha + \int_{x_0 - \delta}^{x_0 + \delta} |f| d\alpha + \int_{x_0 + \delta}^b |f| d\alpha
\]

\[
\geq \int_{x_0 - \delta}^{x_0 + \delta} |f| d\alpha
\]

\[
\geq \int_{x_0 - \delta}^{x_0 + \delta} \frac{1}{2} f(x_0) d\alpha
\]

\[
= \frac{1}{2} f(x_0) (\alpha(x_0 + \delta) - \alpha(x_0 - \delta))
\]

\[
> 0
\]

where the last inequality follows as \( \alpha \) is strictly increasing anf \( f(x_0) > 0 \). Hence, if \( \int_a^b |f| d\alpha = 0 \) one must have \( f \equiv 0 \) identically.
4. (10 points) Let $E_n = [4^{-n}, 4^{-n+\frac{1}{2}}]$. Show that $E = \bigcup_{n=0}^{\infty} E_n$ is (Lebesgue) measurable and compute $m(E)$.

First of all as each $E_n$ is a closed bounded interval, each one is measurable. The $\sigma$-algebra property of measurable sets implies that $E$ is also measurable. As the Lebesgue measure of any interval is its length we have

$$m(E_n) = 4^{-n+\frac{1}{2}} - 4^{-n} = 2 * 4^{-n} - 4^{-n} = 4^{-n}.$$ 

Finally, one readily sees $E_n \cap E_m = \emptyset$ when $n \neq m$. Hence, by the countable additivity property of measurable sets one has

$$m(E) = \sum_{n=0}^{\infty} m(E_n) = \sum_{n=0}^{\infty} 4^{-n} = \frac{4}{3}.$$
5. (15 points) Let $E, F \subset \mathbb{R}$ be (Lebesgue) measurable sets. Show that $m(E \cup F) + m(E \cap F) = m(E) + m(F)$. Explain why this may not hold if $E$ and $F$ are not Lebesgue measurable.

First observe that if $m(E) = \infty$ of $m(F) = \infty$, then as $E, F \subset E \cup F$, $m(E \cup F) = \infty$. That is both left and right hand side are infinity and the result holds.

Hence, we may assume $m(E), m(F) < \infty$. As $E$ and $F$ are measurable one has $E \cap F$ measurable and also $E' = E \setminus (E \cap F)$ and $F' = F \setminus (E \cap F)$. As $E$ is the disjoint union of $E \cap F$ and $E'$ and $F$ is the disjoint union of $E \cap F$ and $F'$ and everything is measurable, countable additivity gives

$$m(E) + m(F) = m(E') + m(E \cap F) + m(F') + m(E \cap F).$$

However, $E \cup F$ is clearly the disjoint union of $E', F'$ and $E \cap F$ and so countable additivity again gives

$$m(E \cup F) = m(E') + m(F') + m(E \cap F).$$

Combining these two observations gives

$$m(E) + m(F) = m(E \cup F) + m(E \cap F).$$

The result need not be true (for outer measure) if $E$ and $F$ are not measurable as we used countable additivity in a crucial way. Indeed, there are disjoint non-measurable sets $E$ and $F$ so that $m^*(E \cap F) + m^*(E \cup F) = m^*(E \cup F) < m^*(E) + m^*(F)$.