Solutions Exam 1 — Mar. 14, 2018

1. (15 points) Let $f : [0,4] \to \mathbb{R}$ increase from -1 to 0 on $[0,1]$ and decrease from 1 to -1 on $(1,4]$ (i.e., $f(0) = -1$, $f(1) = 0$, $\lim_{x\to 1^+} f(x) = 1$, $f(4) = -1$ and f is monotone on $[0,1]$ and $(1,4]$). Show that f is a BV function and compute its total variation $V_0^4 f$.

First observe that, as $\lim_{x\to 1^+} f(x) = 1$, for any $\epsilon > 0$ there is a $\delta > 0$ so if $x \in (1, 1+\delta)$, then $f(x) \geq 1 - \epsilon > 0$. Now let $P = \{x_0, \ldots, x_n\}$ be any partition of $[0,4]$. Up to taking a a finer partition, $P\epsilon$ one may assume that there are is an index $n > i_1 > 0$ so $x_{i_1} = 1$ and so $1 \geq f(x_{i_1+1}) \geq 1 - \epsilon > 0$. Notice that f is increasing on $[x_0, x_{i_1}]$ and is decreasing on $(x_{i_1}, 1]$. Hence,

$$\begin{aligned} V(f,P) \leq V(f,P_\epsilon) &= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=1}^{i_1} (f(x_i) - f(x_{i-1})) + |f(x_{i_1+1}) - f(x_i)| - \sum_{i=i_1+2}^{n} (f(x_i) - f(x_{i-1}) \end{aligned}$$

As two of these sums are telescoping one has

$$V(f,P_\epsilon) = f(x_{i_1}) - f(x_0) + f(x_{i_1} + 1) - f(x_n) + |f(x_{i_1} + 1) - f(x_i)|.$$

As $f(x_{i_1}) = 0$ and $1 \geq f(x_{i_1+1}) \geq 1 - \epsilon$ one has

$$V(f,P_\epsilon) = f(x_{i_1}) - f(x_0) + f(x_{i_1} + 1) - f(x_n) + f(x_{i_1} + 1) - f(x_i)$$

and hence

$$4 - 2\epsilon \leq V(f,P_\epsilon) \leq 4.$$

This implies that

$$V_0^4 f = \sup\{V(f,P) : P\} = 4$$

and so f is BV.

2. (a) (10 points) Give an example of a bounded function $f : [0,1] \to \mathbb{R}$ that is not Riemann integrable on $[0,1]$, i.e. so $f \notin \mathcal{R}[0,1]$. Remember to justify your answer.

The function

$$f(x) = \left\{ \begin{array}{ll} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & x \in [0,1] \backslash \mathbb{Q} \end{array} \right\}$$

is bounded, but is not Riemann integrable. This is because for any partition P of $[0,1]$ one has $U(f,P) = 1$ while $L(f,P) = 0$.

(b) (10 points) Let $\alpha : [-1,1] \to \mathbb{R}$ be given by

$$\alpha(x) = \left\{ \begin{array}{ll} -1 & x \in [-1,0] \\ 1 & x \in (0,1]. \end{array} \right.$$

Give an example of a bounded function $f : [-1,1] \to \mathbb{R}$ with a finite number of discontinuities that is not Riemann-Stieltjes integrable with respect to α, i.e., so $f \notin \mathcal{R}_\alpha[-1,1]$. Remember to justify your answer.

Let $f(x) = \alpha(x)$. Clearly f has exactly one discontinuity at $x = 0$, but $f \notin \mathcal{R}_\alpha[-1,1]$. The reason for this is if P is a partition of $[-1,1]$ which contains 0 one has $U_\alpha(f,P) = 2$ while $L_\alpha(f,P) = -2$.

(c) (15 points) Show that if $f : [-1, 1] \to \mathbb{R}$ is continuous, then is Riemann-Stieltjes integrable with respect to the α from part b), i.e., $f \in \mathcal{R}_\alpha[-1, 1]$. Compute $\int_{-1}^{1} f d\alpha$.

Given $\epsilon > 0$ pick $\delta > 0$ so for $|x| \leq \delta$, $|f(x) - f(0)| \leq \frac{\epsilon}{2}$. Pick any partition P of $[-1, 1]$. Up to passing to a finer partition, P_ϵ, one may assume that $0 = x_{i_1} \in P_\epsilon$ and and $x_{i_1} < x_{i_1+1} < \delta$. Notice that for $i = i_1$ one has $\alpha(x_{i+1}) - \alpha(x_i) = 2$, while for $i \neq i_1$, $\alpha(x_{i+1}) - \alpha(x_i) = 0$.
On $[x_{i_1}, x_{i_1+1}]$ one has

$$f(0) - \epsilon/2 \leq \inf_{[x_{i_1}, x_{i_1+1}]} f(x) \leq \sup_{[x_{i_1}, x_{i_1+1}]} f(x) \leq f(0) + \epsilon/2.$$

Hence,

$$2f(0) + \epsilon \geq U_\alpha(f, P_\epsilon) \geq L_\alpha(f, P_\epsilon) \geq 2f(0) - \epsilon.$$

As $\epsilon > 0$ is arbitrary, and α is increasing this means that

$$2f(0) \geq \inf_P U_\alpha(f, P) \geq \sup_P L_\alpha(f, P) \geq 2f(0).$$

And so one has equality throughout. This implies $f \in \mathcal{R}_\alpha[-1, 1]$ and

$$\int_{-1}^{1} f(t) dt = 2f(0).$$

3. (a) (10 points) Show by example that it is not true that if α is bounded on $[a, b]$ and f is continuous on $[a, b]$, then $\int_a^b |f| d\alpha = 0$ implies f identically vanishes. Remember to justify your answer.

Let α be the weight from problem 2 b). Let $f(x) = x^2$ so f is continuous and $|f| = f$. By the computation of problem 2 c) one has

$$\int_{-1}^{1} |f| d\alpha = |f(0)| = 0$$

but clearly f does not identically vanish.

(b) (15 points) Show that if $\alpha : [a, b] \to \mathbb{R}$ is *strictly* increasing (i.e., $x > y \Rightarrow \alpha(x) > \alpha(y)$), and $f : [a, b] \to \mathbb{R}$ is continuous, then $\int_a^b |f| d\alpha = 0$ implies f identically vanishes.

Suppose $f(x_0) \neq 0$ at some point $x_0 \in [a, b]$. By the continuity of f we may suppose $x_0 \in (a, b)$ and up to replacing f by $-f$ we may assume $f(x_0) > 0$. The continuity of f implies that there is a $\delta > 0$ so $I = [x_0 - \delta, x_0 + \delta] \subset [a, b]$ and $f(x) \geq \frac{1}{2} f(x_0) > 0$ on I. As α is increasing and $|f(t)| \geq 0$ we have, for any $a \leq x < y \leq b$ that

$$\int_x^y |f| d\alpha \geq 0.$$

Using properties of the integral, we have

$$\begin{aligned}
\int_a^b |f| d\alpha &= \int_a^{x_0-\delta} |f| d\alpha + \int_{x_0-\delta}^{x_0+\delta} |f| d\alpha + \int_{x_0+\delta}^{b} |f| d\alpha \\
&\geq \int_{x_0-\delta}^{x_0+\delta} |f| d\alpha \\
&\geq \int_{x_0-\delta}^{x_0+\delta} \frac{1}{2} f(x_0) d\alpha \\
&= \frac{1}{2} f(x_0) \left(\alpha(x_0 + \delta) - \alpha(x_0 - \delta)\right) \\
&> 0
\end{aligned}$$

where the last inequality follows as α is strictly increasing anf $f(x_0) > 0$. Hence, if $\int_a^b |f| d\alpha = 0$ one must have $f = 0$ identically.

4. (10 points) Let $E_n = [4^{-n}, 4^{-n+\frac{1}{2}}]$. Show that $E = \bigcup_{n=0}^{\infty} E_n$ is (Lebesgue) measurable and compute $m(E)$.

First of all as each E_n is a closed bounded interval, each one is measurable. The σ-algebra property of measurable sets implies that E is also measurable. As the Lebesgue measure of any interval is its length we have

$$m(E_n) = 4^{-n+\frac{1}{2}} - 4^{-n} = 2 * 4^{-n} - 4^{-n} = 4^{-n}.$$

Finally, one readily sees $E_n \cap E_m = \emptyset$ when $n \neq m$. Hence, by the countable additivity property of measurable sets one has

$$m(E) = \sum_{n=0}^{\infty} m(E_n) = \sum_{n=0}^{\infty} 4^{-n} = \frac{4}{3}.$$

5. (15 points) Let $E, F \subset \mathbb{R}$ be (Lebesgue) measurable sets. Show that $m(E \cup F) + m(E \cap F) = m(E) + m(F)$. Explain why this may not hold if E and F are not Lebesgue measurable.

First observe that if $m(E) = \infty$ of $m(F) = \infty$, then as $E, F \subset E \cup F$, $m(E \cup F) = \infty$. That is both left and right hand side are infinity and the result holds.

Hence, we may assume $m(E), m(F) < \infty$. As E and F are measurable one has $E \cap F$ measurable and also $E' = E \backslash (E \cap F)$ and $F' = F \backslash (E \cap F)$. As E is the disjoint union of $E \cap F$ and E' and F is the disjoint union of $E \cap F$ and F' and everything is measurable, countable additivity gives

$$m(E) + m(F) = m(E') + m(E \cap F) + m(F') + m(E \cap F).$$

However, $E \cup F$ is clearly the disjoint union of E', F' and $E \cap F$ and so countable additivity again gives

$$m(E \cup F) = m(E') + m(F') + m(E \cap F)$$

Combining these two observations gives

$$m(E) + m(F) = m(E \cup F) + m(E \cap F).$$

The result need not be true (for outer measure) if E and F are not measurable as we used countable additivity in a crucial way. Indeed, there are disjoint non-measurable sets E and F so that $m^*(E \cap F) + m^*(E \cup F) = m^*(E \cup F) < m^*(E) + m^*(F)$.