Problem #1. Let $M$ be an $n$-dimensional differentiable manifold. Define the space of frames of $M$ at $p$, $F_p M$, to be the set of ordered bases of $T_p M$—i.e.,

$$F_p M = \{(v_1, \ldots, v_n) : v_i \in T_p M \text{ and } \{v_1, \ldots, v_n\} \text{ is a basis of } T_p M\}.$$ 

Let $\{F_p M \}_p$ be the frame bundle of $M$. Define the following equivalence relation on $FM = \bigcup_{p \in M} F_p M$, $(p, (v_1, \ldots, v_n)) \sim (q, (w_1, \ldots, w_n)) \iff p = q$ and $v_i = \sum a_{ij} w_j$ where $a_{ij} \in GL(n, \mathbb{R})$ has positive determinant. Let $\bar{M} = FM/\sim$ be the set of equivalence classes.

a) Prove that $FM$ has a natural differentiable manifold structure and that the map $\Pi : FM \to M$ given by $(p, (v_1, \ldots, v_n)) \mapsto p$ is smooth.

b) Show that $\bar{M}$ has a natural differentiable manifold structure and that the map $\pi : \bar{M} \to M$ defined by $[p, (v_1, \ldots, v_n)] \mapsto p$ is a local diffeomorphism.

c) Show that $\bar{M}$ is an orientable $n$-dimensional manifold.

d) Show that $\bar{M}$ is connected if and only if $M$ is connected and non-orientable. In this case, $\bar{M}$ is called the oriented double cover of $M$.

Problem #2. For $X \in \mathcal{X}(M)$, show that if $X_p \neq 0$, then there is a chart $(U, V, \phi)$ around $p$ so that on $U$, $X = \frac{\partial}{\partial x^1}$. Show by example that if $Y \in \mathcal{X}(M)$ is another vector field with $Y_p$ linearly independent from $X_p$, then it may not be possible to find a chart $(U, V, \phi)$ so that on $U$,

$$X = \frac{\partial}{\partial x^1} \quad \text{and} \quad Y = \frac{\partial}{\partial x^2}. $$

Can you give a necessary condition on $X$ and $Y$ for the existence of such a chart?

Problem #3. Let $M$ be a differentiable manifold and $X, Y \in \mathcal{X}_0(M)$ be two vector fields with compact support. If $\phi_t : M \to M$ is the time $t$ flow of $X$ and $\psi_t$ is the time $t$ flow of $Y$ show that

$$\frac{d}{dt}|_{t=0} \gamma_t(p) = [X, Y](p).$$

where

$$\gamma_t = \psi_{-\sqrt{t}} \circ \phi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \phi_{\sqrt{t}}.$$

Problem #4. Show that for vector fields $X, Y_1, \ldots, Y_k \in \mathcal{X}(M)$ and $A \in \Gamma(M, T^{(0,k)} M)$,

$$(L_X A)(Y_1, \ldots, Y_k) = X \cdot (A(Y_1, \ldots, Y_n)) - \sum_{i=1}^k A(Y_1, \ldots, Y_{i-1}, [X, Y_i], Y_{i+1}, \ldots, Y_k).$$

Problem #5. Show the following:

a) If $U$ and $V$ are open subsets of $\mathbb{R}^n$ and $\phi : U \to V$ a diffeomorphism, then when $q = \phi(p)$,

$$(\phi^* dx^1 \wedge \ldots \wedge dx^n)_p = (\det J_p(\phi)) dx_p^1 \wedge \ldots \wedge dx_p^n.$$

b) If $U \subset \mathbb{R}^n$ is an open set, then

$$\Lambda^n U = \{ f dx^1 \wedge \ldots \wedge dx^n : f \in C^\infty(U) \}.$$ 

That is, $\Lambda^n U$ is the trivial rank one vector bundle.

c) For connected $M$, $\Lambda^n M$ is a rank one vector bundle that is trivial if and only if $M$ is orientable.