Problem #1. Let \((M, g)\) and \((N, h)\) be Riemannian manifolds and \(f\) a smooth positive function on \(M\). Define a \(0,2\)-tensor on \(M \times N\) by
\[
(g \times_f h)_p(X_p, Y_p) = g_{\pi_M(p)}(T_p \pi_M(X_p), T_p \pi_M(Y_p)) + f^2(\pi_M(p)) h_{\pi_N(p)}(T_p \pi_N(X_p), T_p \pi_N(Y_p))
\]
where here \(\pi_N, \pi_M\) are the natural projections.

a) Show that \(g \times_f h\) is a Riemannian metric on \(M \times N\). It is called a warped product metric.
b) Let \((M, g) = (\mathbb{R}^+, g^E)\) and \((N, h) = (\mathbb{S}^n, g^S)\) and denote by \(r \in C^\infty(M)\) the standard coordinate on \(\mathbb{R}^+\). Consider the following family of warped product metrics
\[
c_\lambda = g \times_{\lambda r} h.
\]
Show that \((\mathbb{R}^+ \times \mathbb{S}^1, c_\lambda)\) is locally isometric to \((\mathbb{R}^2 \setminus \{0\}, g^E)\) for all \(\lambda > 0\).
c) Show that \((\mathbb{R}^+ \times \mathbb{S}^1, c_\lambda)\) is isometric to \((\mathbb{R}^2 \setminus \{0\}, g^E)\) if and only if \(\lambda = 1\). (Hint: Consider parallel transportation around \(\pi_{\mathbb{R}^+}(1)\)).

Problem #2.

a) Let \((M, g) = ((0, 2\pi), g^E)\) and \((N, h) = (\mathbb{S}^n, g^S)\) show that
\[
g^E \times_{\sin(r)} g^S
\]

is locally isometric to \((\mathbb{S}^{n+1}, g^S)\).
b) Let \((M, g) = (\mathbb{R}^+, g^E)\) and \((N, h) = (\mathbb{S}^n, g^S)\) show that
\[
g^E \times_{\sinh(r)} g^S
\]

is locally isometric to \((\mathbb{H}^{n+1}, g^H)\).

Problem #3. Verify that \(\phi \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)\) is an isometry of \((\mathbb{R}^n, g^E)\) if and only if \(\phi\) can be identified with an element of \(O(n) = \{A \in \mathbb{R}^{n \times n} : A^T A = I_n\}\).

Problem #4. Let \((M, g)\) be a closed Riemannian manifold. We say a vector field \(X \in \mathcal{X}(M)\) is a Killing field, provided \(\phi_t^* g = g\), for all \(t\), where \(\phi_t\) is the time \(t\) flow of \(X\). Show that \(X\) is Killing if and only if for all vector fields \(X, Y \in \mathcal{X}(M)\)
\[
g(\nabla_X Y, Z) = -g(\nabla_Z X, Y).
\]
Here \(\nabla\) is the Levi-Civita connection of \(g\). Hint: Use that \(X\) is Killing if and only if \(L_X g = 0\).

Problem #5. Define a connection \(\nabla\) on \(\mathbb{R}^3\) by
\[
\nabla_{\partial_{x_1}} \partial_{x_2} = \nabla_{\partial_{x_2}} \partial_{x_1} = \nabla_{\partial_{x_3}} \partial_{x_2} = \nabla_{\partial_{x_3}} \partial_{x_3} = 0.
\]
where here \(\partial_{x_i} = \frac{\partial}{\partial x_i}\) are the coordinate vector fields of the standard coordinates.

a) Verify that this connection is compatible with \(g^E\) but is not torsion-free.
b) Determine the zero-acceleration curves of \(\nabla\).
c) Determine the parallel transport of \(\nabla\) along zero-acceleration curves.