Problem #1. Let \((M,g)\) be a Riemannian manifold and \(p \in M\). Suppose that \((x^1,\ldots,x^n)\) are geodesic normal coordinates at \(p\) (so \(x'(p) = 0\)). Show that in these coordinates metric \(g\) satisfies the estimate

\[ |g_{ij}(q) - \delta_{ij}| \leq C_1 r^2(q) \]

where \(q\) is near \(p\) and \(C_1 > 0\) depends on \(g\). Here

\[ r(p) = \sqrt{(x^1(p))^2 + \ldots + (x^n(p))^2}. \]

is the euclidean distance to the origin in the coordinate chart. That is, in geodesic normal coordinates the euclidean metric approximates the metric \(g\) to first order.

Problem #2. Let \((M,g)\) be a Riemannian manifold and \(p \in M\). Use the preceding result to show that for \(r > 0\) sufficiently small

\[ |Vol_g(B_r(p)) - \omega_n r^n| \leq C_2 r^{n+2}. \]

Here \(B_r(p)\) is the geodesic ball of radius \(r\), \(\omega_n\) is the volume of the unit ball in \(\mathbb{R}^n\) and \(C_2 > 0\) depends on \(g\). You should use the fact that \(\det(I_n + sA) = 1 + s \text{tr}A + O(s^2)\).

Problem #3. Consider \((S^n,g^S)\). Arguing geometrically, show that if \(n \geq 2\), then for all \(r > 0\)

\[ Vol_g^S(B_r(p)) < \omega_n r^n. \]

What happens when \(n = 1\)?

Problem #4. Let \((M,g)\) and \((N,h)\) be Riemannian manifolds. Show that if \(\phi : M \to N\) is a Riemannian isometry (i.e., it is a diffeomorphism and \(\phi^*h = g\)), then it is a metric isometry (i.e., \(d_h(\phi(p),\phi(q)) = d_g(p,q)\)).

Problem #5. Let \((M,g)\) be a complete non-compact Riemannian manifold. Prove that for each \(p \in M\), there is a ray \(\gamma : [0,\infty) \to M\) starting from \(p\). That is, \(\gamma(0) = p\) and \(\gamma\) is a minimal geodesic.