Math 645, Fall 2017: Assignment #6

Due: Thursday, October 26th

Problem #1. Verify that the expression

\[ K_p(x, y) = \frac{R_p(x, y, x, y)}{|x|^2|y|^2 - g_p(x, y)^2} \]

depends only on the two-plane in \( T_p M \) spanned by \( x, y \in T_p M \).

Problem #2. Prove the 2nd Bianchi identity:

\[ (D_T R)(X, Y, Z, W) + (D_Z R)(X, Y, W, T) + (D_W R)(X, Y, T, Z) = 0. \]

Hint: Compute using the geodesic frame introduced in Problem 5 of Homework 4.

Problem #3. Use the 2nd Bianchi identity to show the following (known as Schur’s theorem): If \((M, g)\) is an \( n \)-dimensional manifold with \( n \geq 3 \) and for all \( p \in M \), \( K_p(\sigma) = K(p) \) for each two-plane \( \sigma \subset T_p M \) (i.e., the sectional curvature is independent of the two-plane and depends only on the point), then \((M, g)\) has constant curvature (i.e., the sectional curvature is also independent of the point of \( M \)). Hint: The hypotheses means that \( R_p = K(p)R^1_p \), where \( R^1_p \) was defined in class. As \( D_X R^1 = 0 \), this means \( D_X R = (X \cdot K)R^1 \).

Problem #4. Show that if \((M, g)\) is a three dimensional Riemannian manifold, then the Ricci tensor uniquely determines the Riemann curvature tensor. Hint: This is a purely algebraic fact.

Problem #5. Show that if \((M, g)\) has vanishing Riemann curvature tensor, then the \( \text{exp}_p : T_p M \to M \) is a local isometry between \((T_p M, g_p)\) and \((M, g)\).