Math 645, Fall 2017: Assignment #8

Due: Tuesday, November 14th

Problem #1. Let $\phi \in C^{\infty}([0,L])$ satisfy $\phi > 0$, $\phi^{(2k)}(0) = 0$ for all $k \geq 0$ (i.e., the derivatives of $\phi$ at 0 behave like those of an odd function) and $\phi'(0) = 1$ and consider the warped product metric $(M', g') = ((0,L) \times \mathbb{S}^{n}, g^{E} \times g^{S})$.

a) Show that there is a $n+1$ dimensional Riemannian manifold $(M, g)$ and an isometric embedding $\phi : M' \to M$ so that $\lim_{r \to 0} \phi(r, v_{i}) = p_{0} \in M$ exists and $f(M') = M \setminus \{ p_{0} \}$.

b) Determine what the geodesics emanating from $p_{0}$ correspond to in $M'$.

c) Compute the sectional curvatures of $(M, g)$ in terms of $\phi$. (Hint: The Jacobi equation along geodesics emanating from $p_{0}$ – treat $p_{0}$ separately.)

Problem #2. Let $(M, g)$ be a complete Riemannian manifold of dimension $n$ with non-positive curvature. Let $X$ be a Killing vector field on $M$.

a) Show that if $X$ has two distinct zeros, then $X$ must vanish on any geodesic joining the two zeros. (Hint: Use last weeks homework.)

b) (Bonus): Show that when $n = 2$, if $X$ admits two distinct zeros, then it must vanish identically. (Hint: Use the identity from Homework 3 and the result from last week)

c) (Bonus): Show by example that $X$ may have two distinct zeros but not vanish identically when $n \geq 3$.

Problem #3. Fix a manifold $M$. Two metrics $g$ and $h$ on $M$ are conformal provided there is a function $u \in C^{\infty}(M)$ so that $h = e^{2u}g$.

a) If $D^{h}$ is the Levi-Civita connection of $h$ and $D^{g}$ is the Levi-Civita connection of $g$ show that for any $X, Y \in \mathfrak{X}(M)$

$$D^{h}_{X}Y = D^{g}_{X}Y + (X \cdot u)Y + (Y \cdot u)X - g(X, Y)\nabla_{g}u.$$  

Here $\nabla_{g}u$ is the $g$-gradient of $u$. (Hint: use compatibility with the metric).

b) Use this formula to determine the geodesics of the upper half-plane model of hyperbolic space. Recall, this is the Riemannian manifold $(\mathbb{H}^{n}, g^{H})$ where $\mathbb{H}^{n} = \{ x^{n} > 0 \} \subset \mathbb{R}^{n}$ and with metric $g^{H} = (x^{n})^{-2}g^{E}$.

Problem #4. Given two Riemannian manifolds $(M, g)$ and $(N, h)$ we say a map $f : M \to N$ is conformal if $f^{*}h$ is conformal to $g$.

a) Show that the map $f : U \to \mathbb{R}^{n}$ where $U \subset \mathbb{R}^{n}$ given by

$$f(x) = p_{0} + \frac{\lambda A(x - p_{1})}{|x - p_{1}|^{\epsilon}}$$

is conformal from $(U, g^{E})$ to $(\mathbb{R}^{n}, g^{E})$ when $\lambda > 0$, $p_{0} \in \mathbb{R}^{n}$, $p_{1} \in \mathbb{R}^{n} \setminus U$, $\epsilon = 0, 2$ and $A \in O(n)$. That is compositions of translation, rotation, homothetic scaling and “inversion” are conformal.

b) Use complex analysis to give a conformal map not of this form when $n = 2$.

c) (Bonus): Show that when $n \geq 3$ the only conformal maps are those found in a). This is called Liouville’s theorem.

Problem #5. Show that if $(\mathbb{R}^{2}, g)$ is a complete Riemannian manifold, then

$$\lim_{r \to \infty} \inf_{(x^{1})^{2} + (x^{2})^{2} \geq r^{2}} S(x^{1}, x^{2}) \leq 0.$$  

Here $(x^{1}, x^{2})$ are the standard coordinates on $\mathbb{R}^{2}$ and $S(x^{1}, x^{2})$ is the scalar curvature at the point $(x^{1}, x^{2})$. 