Math 645, Fall 2017: Assignment #9
Due: Thursday, December 7th

Problem #1. Prove Wirtinger’s inequality: If \( f : [0, \pi] \to \mathbb{R} \) is \( C^2 \) and satisfies \( f(0) = f(\pi) = 0 \), then
\[
\int_0^\pi f^2 \, dt \leq \int_0^\pi (f')^2 \, dt,
\]
with equality if and only if \( f(t) = c \sin(t) \) where \( c \in \mathbb{R} \). Hint: Use Fourier series or the calculus of variations.

Problem #2. Let \( M \) be a complete simply connected Riemannian manifold. For each \( p \in M \), denote by \( \text{conj}(p) \), the set of first conjugate points of \( p \) (i.e., for each \( v \in T_p M \) with \( v \neq 0 \), consider the first conjugate point of the geodesic \( c_v(s) = \exp_p(sv) \)).

a) Show that if \( p' \in \text{conj}(p) \) satisfies \( d_g(p,p') = L \), then there is a unit speed geodesic \( \gamma : [0,L] \to M \) connecting \( p \) to \( p' \), so that there is a non-trivial Jacobi field \( J \) along \( \gamma \) so \( J(0) = J(L) = 0 \), and \( I(J,J) = 0 \).

b) If for each \( p \in M \), \( \text{conj}(p) \) consists of one point \( p' \) with \( d_g(p,p') = d_g(p, \text{conj}(p)) = \pi \) and the sectional curvature of \( M \) satisfies \( 0 < \delta \leq K \leq 1 \), then \( M \) is isometric to \( (\mathbb{S}^n, g^\mathbb{S}) \). Hint: Consider the geodesic, \( \gamma \), Jacobi field, \( J \), from part a), using the Jacobi equation, the index property of \( J \) and the previous problem conclude that along the geodesic \( \gamma \) one has \( K(\gamma', J) = 1 \), where \( K(\gamma', J) \) is the sectional curvature of the two plane spanned by \( \gamma' \) and \( J \).

Problem #3. Let \( (M,g) \) be a Riemannian manifold. Let \( \Omega \subset M \) be an open domain which is strongly convex (i.e. for every two points \( p,q \in \Omega \) there is a minimizing geodesic contained in \( \Omega \) connecting \( p \) to \( q \)) and so that \( \partial \Omega \) is a smooth submanifold.

a) Show that the second fundamental form \( \Pi_N \) of \( \partial N = \partial \Omega \) with respect to the outward pointing normal to \( \Omega \) is non-negative in that \( \Pi_N^p(v,v) \geq 0 \) for all \( v \in T_p \partial \Omega \) and \( p \in \partial \Omega \).

b) The domain \( \Omega \) in \( M \) is convex if \( \partial \Omega \) is a submanifold curve whose whose second fundamental form with respect to the outward pointing normal to \( \Omega \) is non-negative. Show by example that a domain may be convex but fail to be strongly convex.

Problem #4. Show that if \( M \subset \mathbb{R}^{n+1} \) is a compact hypersurface (i.e., a codimension one submanifold), then there is a point \( p \in M \) so that the second fundamental form of \( M \) is strictly positive (with respect to some choice of unit normal). Hint: Consider the smallest euclidean ball centered at the origin containing \( M \).

Problem #5. Show that there can be no \( C^2 \) isometric embedding \( f : (\mathbb{S}^2, g^T) \to (\mathbb{R}^3, g^E) \).

Hint: Use the previous exercise.