110.108: Calculus I
Solutions to Exam 2

1. Evaluate the following integrals. Show your work.

a) $\int \sin x - 3x^2 \, dx = -\cos x - x^3 + c$

b) $\int_0^3 x\sqrt{1+x}\, dx$

Use substitution with $u = 1 + x$. Then $\frac{du}{dx} = 1$ and $x = u - 1$. When $x = 0$, $u = 1 + 0 = 1$, and when $x = 3$, $u = 1 + 3 = 4$. Thus,

$$\begin{aligned}
\int_0^3 x\sqrt{1+x}\, dx &= \int_1^4 (u-1)\sqrt{u}\, du \\
&= \int_1^4 u^{3/2} - u^{1/2}\, du \\
&= 2/5u^{5/2} - 2/3u^{3/2}|_1^4 \\
&= 2/5(32) - 2/3(8) - 2/5 + 2/3 \\
&= 116/15.
\end{aligned}$$

2. (*12 points*) Let F and G be the functions defined by

$$F(x) = \int_1^x \sin^2 2t\, dt, \qquad G(x) = \int_1^{x^3} \sin^2 2t\, dt.$$

Use these functions to find the following.

a) $F(1) = \int_1^1 \sin^2 2t\, dt = 0$ by a property of the definite integral.

b) $F'(x) = \frac{d}{dx}\int_1^x \sin^2 2t\, dt = \sin^2 2x$ by a theorem proved in class.

c) $G'(x)$

Note that $G(x) = F(x^3)$. By the chain rule,

$$G'(x) = F'(x^3) \cdot 3x^2 = (\sin^2 2x^3)(3x^2).$$

3. (*8 points*) State the Fundamental Theorem of Calculus.

If f is continuous on $[a, b]$ and G is an antiderivative of f, then

$$\int_a^b f(t)\,dt = G(b) - G(a).$$

4. (*20 points*) Set up, BUT DO NOT EVALUATE, integrals to calculate the following.

a) The area bounded by the curves $y = x^2 - 4$ and $y = 2x - 1$.

$$\text{area} = \int_{-1}^{3} (2x - 1) - (x^2 - 4)\,dx.$$

b) The volume of the solid obtained by revolving the region bounded by $y = x^2$ and $y = 4$ about the line $x = -2$.

$$\text{volume} = \pi \int_0^4 (\sqrt{y} + 2)^2 - (-\sqrt{y} + 2)^2\,dy$$

5. (*24 points*) Do the following for the function $f(x) = \dfrac{x}{(x+3)^2}$. Use the facts that $f'(x) = \dfrac{3-x}{(x+3)^3}$ and $f''(x) = \dfrac{2x-12}{(x+3)^4}$. Show all work.

a) Find the horizontal and vertical asymptotes (if any) of f.

The function f has a vertical asymptote of $x = -3$ and a horizontal asymptote of $y = 0$.

b) Find the intervals where f is increasing and decreasing.

$f'(x)$ is undefined when $x = -3$ and is 0 when $x = 3$. On the interval $(-\infty, -3)$, $f'(x) < 0$. On the interval $(-3, 3)$, $f'(x) > 0$. On the interval $(3, \infty)$, $f'(x) < 0$. Thus, f is decreasing on $(-\infty, -3)$ and $(3, \infty)$, and f is increasing on $(-3, 3)$.

c) Find the intervals where f is concave up and concave down.

$f''(x)$ is undefined when $x = -3$ and is 0 when $x = 6$. On the intervals $(-\infty, -3)$ and $(-3, 6)$, $f''(x) < 0$. On the interval $(3, \infty)$, $f''(x) > 0$. Thus, f is concave down on $(-\infty, -3)$ and $(-3, 6)$, and is concave up on $(6, \infty)$.

d) Use the information in a)-c) to sketch the graph of f. To receive full credit your graph must exhibit all features from a)-c).

This will be done in class on Monday, November 25th.

6. (*20 points*) A piece of string 20 meters long is cut into two pieces. One piece will form a square and the other piece will form a circle. Show all of your work and justify your answers in doing the following.

a) Find the length of the pieces that would maximize the total area enclosed by the square and the circle.

Let x be the length of a side of the square and r be the radius of the circle. The quantity to be maximized is A, the total area enclosed by the square and the circle. We have

$$A = x^2 + \pi r^2.$$

Since we are using 20 meters of string, we also have

$$4x + 2\pi r = 20.$$

Thus,

$$r = \frac{20 - 4x}{2\pi} = \frac{10 - 2x}{\pi},$$

and

$$A = x^2 + \frac{(10 - 2x)^2}{\pi}.$$

We differentiate A to find its critical points and obtain:

$$A' = 2x - \frac{4}{\pi}(10 - 2x)$$

or

$$A' = \left(\frac{2\pi + 8}{\pi}\right) x - \frac{40}{\pi}.$$

Setting $A' = 0$ and solving for x we find the critical number

$$x = \frac{20}{\pi + 4}.$$

Note that the smallest value x can have is 0, when all of the string is used to make the circle. The largest value that x can have is 5 when all of the string is used to make the square. Thus, the domain of A is $[0, 5]$. The maximum value of A occurs at $0, 5$, or the critical number $\frac{20}{\pi+4}$. If we differentiate A' we see that

$$A'' = \frac{2\pi + 8}{\pi} > 0.$$

Since $A'' > 0$, the critical number gives us the minimum value for A. To find the maximum value of A we test the endpoints of the domain of A:

$$A(0) = \frac{100}{\pi}, \quad A(5) = 25.$$

Since $\pi < 4$, $\frac{100}{\pi} > \frac{100}{4} = 25$. Thus the maximum area occurs when 20 meters of string is used for the circle, and none is used for the square.

b) Find the length of the pieces that would minimize the total area enclosed by the square and the circle.

We saw in part a) that the minimum value for A occurs when $x = \frac{20}{\pi+4}$. Thus, the quantity used for the square will be $4x = \frac{80}{\pi+4}$ and the quantity used for the circle will be $20 - \frac{80}{\pi+4} = \frac{20\pi}{\pi+4}$ meters.