Weil–Petersson isometries via the pants complex

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joint w/ Jeff Brock

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Teichmüller space

$S =$ closed, connected, orientable surface
with $\chi(S) < 0$

$\mathsf{T}(S) =$ Teichmüller space for S
$= \{(X, f)\}/\sim$

- X a hyperbolic surface
- $f : S \to X$ a homeomorphism
- $(X_1, f_1) \sim (X_2, f_2)$ if

  $f_2 \circ f_1^{-1}$ isotopic to an isometry

Weil–Petersson metric

$T_X^\star \cong QD(X)$

Petersson inner product on $QD(X)$:

$$\langle \phi, \psi \rangle = \int_X \frac{\phi\bar{\psi}}{\rho^2}$$

$\rho =$ hyperbolic metric on X

$T_X \cong BD(X)/\sim$

Weil–Petersson metric defined by

$$\langle \mu, \phi \rangle = \int_X \mu\phi$$

for $\mu \in T_X$, $\phi \in T_X^\star$.

Mapping class group

$\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^{\pm}(S))$

$\mathrm{Mod}(S) \circlearrowleft \mathrm{T}(S)$:

$$\phi \cdot (X, f) = (X, f \circ \phi^{-1})$$

Action is by isometries.

Main Theorem

The natural map

$$\eta : \mathrm{Mod}(S) \to \mathrm{Isom}(\mathrm{T}(S))$$

is surjective.

· $\ker(\eta) \cong \mathbb{Z}_2$ for $S \in \{S_{1,1}, S_{1,2}, S_{2,0}\}$

· $\ker(\eta) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $S = S_{0,4}$

· $\ker(\eta) = \mathrm{Mod}(S)$ for $S = S_{0,3}$

· $\ker(\eta) = 1$ otherwise

Masur–Wolf '00: proof for $S \notin \{S_{1,1}, S_{1,2}, S_{0,4}\}$
Wolpert '03: shortened last step of M–W
Brock–Margalit '04: new proof for all S

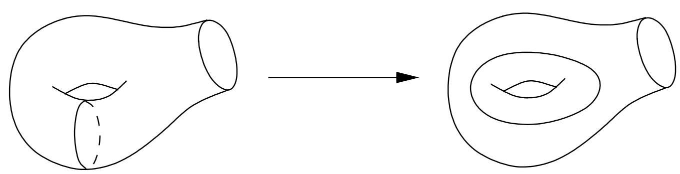
Both proofs follow Ivanov.

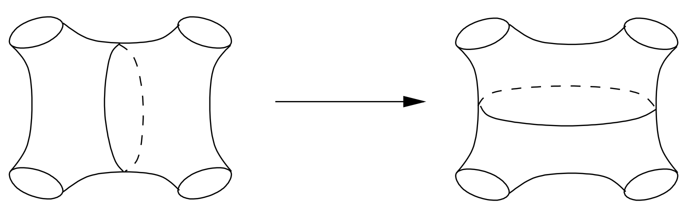
Pants graph P(S)

Hatcher–Thurston '80.

vertices: pants decompositions of S

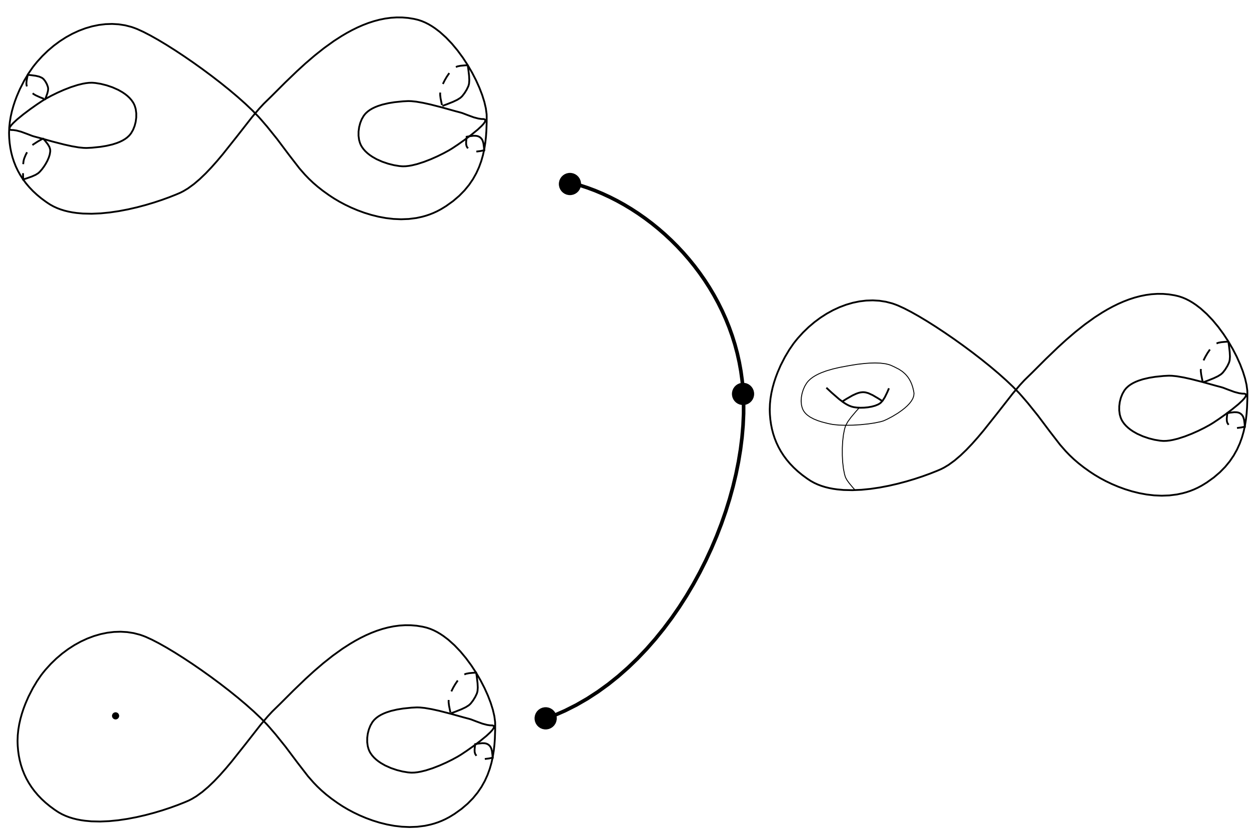
edges:



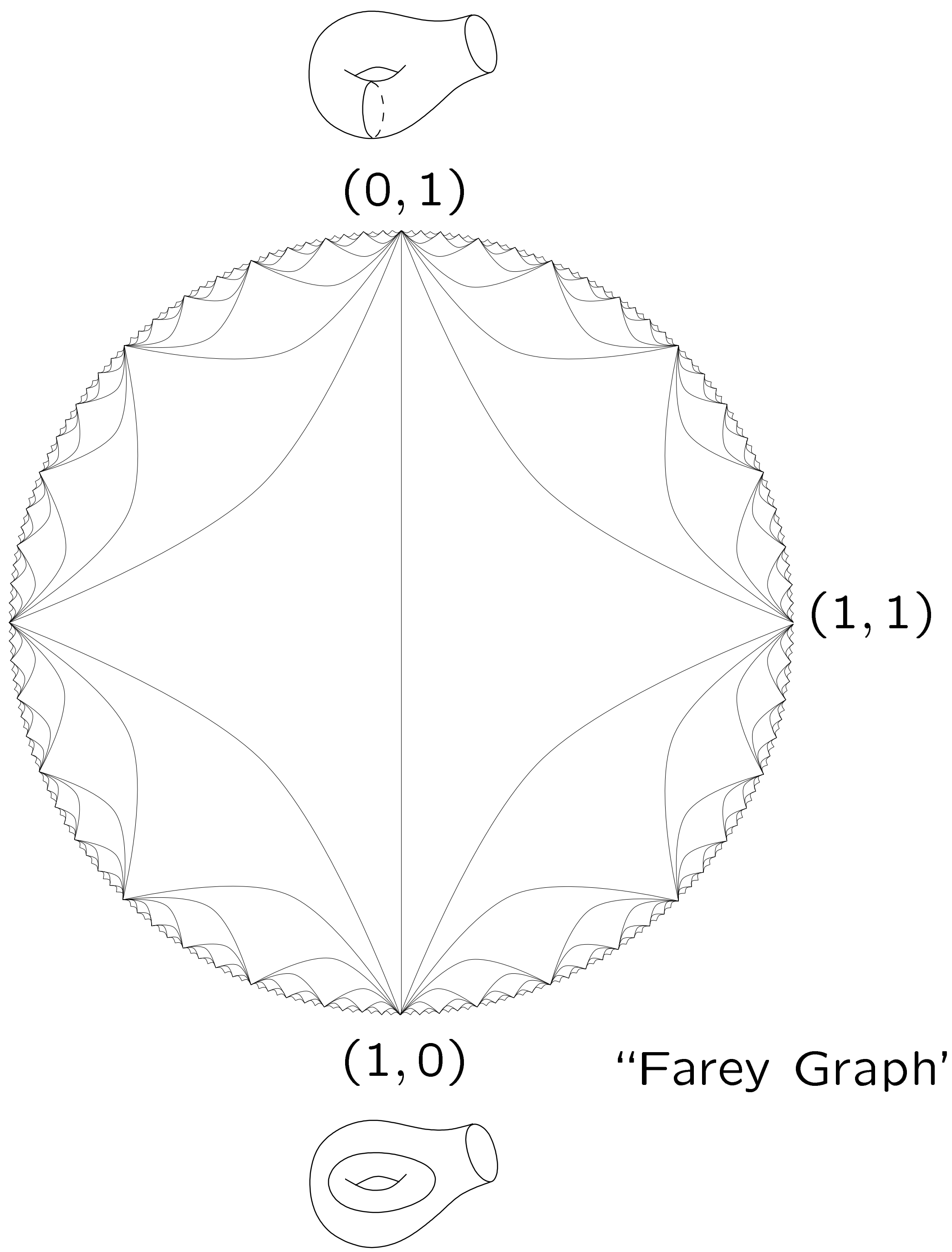


P(S) at infinity in T(S)

- WP not complete (Wolpert '75, Chu '76)
- Completion $\overline{\mathsf{T}(S)}$ obtained by adding *noded surfaces at infinity* (Masur '76)
- $\overline{\mathsf{T}(S)}$ naturally stratified



Edge stratum



Outline of proof

Let $I \in \mathsf{Isom}(\mathsf{T}(S)) \rightsquigarrow \bar{I} \in \mathsf{Isom}(\overline{\mathsf{T}(S)})$.

Step 1: $\bar{I}$ surjective, preserves strata.

Step 2: $\bar{I}$ preserves edges of $\mathsf{P}(S)$
$\rightsquigarrow \bar{I}_\star \in \mathsf{Aut}(\mathsf{P}(S))$.

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Step 3: Apply
Thm (Margalit). *The natural map*

$$\mathsf{Mod}(S) \to \mathsf{Aut}(\mathsf{P}(S))$$

is surjective.

Step 4: Apply
Thm (Brock+Wolpert). *$\bar{I}$ determined by $\bar{I}_\star$.*