AUTOMORPHISMS OF THE FRICKE CHARACTERS OF FREE GROUPS

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Abstract. The space of Fricke characters V_{F_n} of a free group F_n – the set of all characters of representations of F_n into $SL(2,\mathbb{C})$ – is an irreducible affine variety which can be embedded in affine is many ways. Herein, we construct a preferred embedding into the affine space $\mathbb{K}^{2^n-1}$ (for $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$) via a set of distinguished characters detailed by Horowitz. This embedding has the property that the natural $Out(F_n)$ action on V_{F_n} is the homomorphic image via the restriction map of the action of a group of volume preserving polynomial automorphisms on $\mathbb{K}^{2^n-1}$, which we call $POut(F_n)$, the pre-outer automorphism group of F_n. When F_n is the fundamental group of a compact surface S, the subgroup of $POut(F_n)$ whose homomorphic image is the mapping class group of S, $MCG(S)$ consists of polynomial automorphisms of constant unit Jacobian. This property is not shared with the more common affine embedding using the Sibirskii minimal generating set analyzed by, for example, Drensky and Gonzalez-Acuña and Montesinos-Amilibia. For $n < 4$, the homomorphism $\Phi_n : POut(F_n) \to Out(F_n)$ is an isomorphism and $Out(F_n)$ acts directly on the ambient affine space. However, for $n \geq 4$, the kernel $\ker \Phi_n$ is nontrivial. We compute $\ker \Phi_4$ explicitly and show that it is the universal rank three Coxeter group. More generally for $n > 4$, we show $\ker \Phi_n$ is rank-3 with torsion-free generators.

1. Introduction

In this paper, we consider the algebraic model of the set of all special linear characters of the free group on n-letters F_n as the affine variety $V_{F_n,\mathbb{K}} \subset \mathbb{K}^{2^n-1}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ via an embedding which uses the Horowitz trace coordinates as a basis for affine space. $V_{F_n,\mathbb{K}}$ is also the algebro-geometric (categorical) quotient of the set of all representations F_n into $SL(2,\mathbb{C})$ (restricted to the real points, when $\mathbb{K} = \mathbb{R}$), by the conjugacy map: two representations are identified if they are conjugate or if their conjugacy classes are inseparable as points of the usual quotient ("algebro-geometric" refers to the fact that the ring of functions on V_{F_n} identifies with the ring of conjugation-invariant regular functions on the space of all representations). The geometry of V_{F_n} has been extensively studied (see, for example, Goldman [7]), and it is well known that automorphisms of F_n act on V_{F_n} via the outer automorphism group $Out(F_n)$, preserving much of this geometry. For example, $Out(F_n)$ preserves the volume form on V_{F_n} (up to sign). Horowitz [9] showed that the character of any $W \in F_n$ can be written as an integer polynomial in the $2^n - 1$ Horowitz generators. Thus by the Culler and Shalen [1] evaluation map, we use these Horowitz generators as coordinates of affine space and embed $V_{F_n} \subset \mathbb{K}^{2^n-1}$ as a algebraic set (proper, for $n > 2$). The action of individual outer automorphisms on these trace coordinates provides a representation of each free

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group automorphism as a polynomial automorphism of the ambient affine space. That the group action of $Out(F_n)$ on V_{F_n} does not extend in general to the ambient space is a result of McCool [16]. However, the group generated by "lifting" the four Nielsen generators of $Out(F_n)$ does act on the ambient space and has $Out(F_n)$ as a homomorphic image with nontrivial kernel when $n > 3$. We call McCool's lifted group the pre-outer automorphism group $POut(F_n)$, with $\Phi_n : POut(F_n) \to Out(F_n)$ the epimorphism determined by McCool's construction. We show the following:

Theorem 1.1. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $POut(F_n)$ acts as volume-preserving, polynomial automorphisms on $\mathbb{K}^{2^n-1}$ and restricts to the action of $Out(F_n)$ on $V_{F_n,\mathbb{K}} \subset \mathbb{K}^{2^n-1}$.*

Thus, instead of looking for an affine embedding of V_{F_n} where the $Out(F_n)$-action extends to affine space, we have an affine embedding where a group action restricts to the $Out(F_n)$-action on V_{F_n}. Essentially, we prove this by showing that any $\theta \in POut(F_n)$, viewed as a polynomial automorphism of affine space, has constant Jacobian everywhere either 1 or -1. It is worth noting here that, in general, the Horowitz generating set is not a minimal generating set for the trace ring. Sibirskii [21] (See also Drensky [2]) stipulated a minimal generating set of the first $p = \frac{n(n^2+5)}{6}$ Horowitz generators; characters of basic free group words whose word length is 3 or less. This minimal set is often used as a standard way to view the Fricke variety of a free group as an algebraic set: González-Acuña and Montesinos-Amilibia [5] embed V_{F_n} into $\mathbb{C}^p$ and construct a basis for the ideal of functions that vanish on V_{F_n}. However, in general $Out(F_n)$ does not extend as automorphisms in a natural way in this "smaller" affine space. This is the focus of Example 6.4 below.

Let F_n be a surface group – the fundamental group of a compact orientable surface S (necessarily with boundary in this case). Then, depending on the genus of S and the number of boundary components, $V_{F_n,\mathbb{K}}$ possesses the additional structure of a (complex, when $\mathbb{K} = \mathbb{C}$) Poisson space, with (complex) symplectic leaves corresponding to the characters of representations which agree on the boundary components (see Huebschmann [11]; These are the *relative character varieties* of S). The mapping class group of the surface $MCG(S)$ is a subgroup of $Out(F_n)$ (the set of isotopy classes of orientation preserving homeomorphisms of S which pointwise fix the boundary of S). In this case, $\theta \in MCG(S) < Out(F_n)$ preserves this Poisson structure, and acts symplecticly on the leaves. Hence θ preserves symplectic volume on the leaves. One can construct a (complex) Poisson volume on all of $V_{F_n,\mathbb{K}}$ via the symplectic volume on the leaves and a pull-back of a volume on the leaf space (a parameterization of the symplectic leaves formed by evaluation on the boundary components). Since the leaves are fixed by mapping classes, the Poisson volume on V_{F_n} is also preserved. Call the subgroup of $POut(F_n)$, whose homomorphic image under Φ_n is $MCG(S)$, the pre-mapping class group of S, $PMCG(S)$.

Theorem 1.2. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $PMCG(S)$ acts as constant unit Jacobian polynomial automorphisms on $\mathbb{K}^{2^n-1}$ and restricts to the Poisson action of $MCG(S)$ on $V_{S,\mathbb{K}} \subset \mathbb{K}^{2^n-1}$.*

Goldman [8] has calculated the Poisson structure in these trace coordinates for some low-genus surfaces. It would be interesting to know how the standard volume form on $\mathbb{K}^{2^n-1}$ relates to the Poisson volume form on $V_{F_n,\mathbb{K}}$ under this Horowitz embedding, since both are preserved by $PMCG(S)$.

It is known that, for $n < 4$, the kernel of the homomorphism Φ_n is trivial, and hence Φ_n is an isomorphism (See [16]). For $n \geq 4$, $\ker \Phi_n$ is not trivial, and McCool has constructed an example of elements in the kernel, for $n = 4$. Herein, we use the structure of the lifts to calculate $\ker \Phi_4$:

Theorem 1.3. $\ker \Phi_4$ *is the universal, rank-*3 *Coxeter group.*

Theorem 1.4. *for* $n > 4$, $\ker \Phi_n$ *is generated by three torsion-free elements.*

This paper is organized as follows: In Section 2, we review the details of the Horowitz embedding. We use the basic words of a free group to construct the indeterminates of the polynomial ring whose quotient is the character ring of F_n, and the evaluation map on the representation variety whose image is V_{F_n}. Using the presentation of $Out(F_n)$ by Nielsen transformations, we show in Section 3 that these generators all act on $\mathbb{K}^{2^n-1}$ as polynomial automorphisms whose Jacobian has constant determinant 1 or -1 everywhere. Thus a top level differential form is preserved up to sign. This will prove Theorem 1.1, which we detail in Section 4. Theorem 1.2 will then follow once the first result is interpreted appropriately. In Section 5, we calculate the kernel of Φ_4, and find a generating set for $\ker \Phi_n$, $n > 4$. The last section, Section 6, presents some examples for $n = 2$, 3, and 4. Here we explicitly show that this construction does not work in the more common affine embedding of $V_{F_n} \subset \mathbb{C}^p$, $p = \frac{n(n^2+5)}{6}$, using the Sibirskii generators.

2. The Horowitz embedding

Fix an ordered generating set for F_n. A word $X \in F_n$ is *basic* if each letter of X is a generator of exponent one, and each letter of X is strictly greater than the one after it. Evidently, there are exactly $2^n - 1$ basic words in F_n. One can extend this ordering on the generators to the basic set: Shorter words are greater than longer words, and words of the same length are compared based on the letters in the first position that they disagree.

Example 2.1. Let $F_3 = \langle A, B, C\rangle$ ordered such that $A > B > C$. Then the basic set for F_3 can be ordered from highest to lowest

$$\{A, B, C, AB, AC, BC, ABC\}.$$

For $\mathbb{K}$ a field, a special linear character of X (the Fricke character of X) is a $\mathbb{K}$-valued, conjugation invariant map on the set of all $SL(2, \mathbb{K})$-representations of F_n. Denote this map

$$\mathrm{tr}_X : Hom(F_n, SL(2, \mathbb{K})) \to \mathbb{K}, \quad \mathrm{tr}_X(\phi) = \mathrm{tr}(\phi(X)). \tag{2.1}$$

Adopt the notation that for a word $X \in F_n$, we will use an upper case letter, and for its associated character, we will use the lower case equivalent. Thus, x is the character of the word $X \in \mathbb{F}_n$ (that is, $x = tr_X$).

It is a fact proposed by Fricke [4] was proved by Horowitz [9] that the character of any free group word can be expressed as a polynomial with integer coefficients in the $2^n - 1$ characters of the basic words in F_n. These are the Horowitz generators of the set of all characters of F_n. Call this set $\mathcal{H}_n$. In Example 2.1, the Horowitz generating set of the characters of F_3 is

$$\mathcal{H}_3 = \{x_i\}_{i=1}^{7} = \{a, b, c, ab, ac, bc, abc\}.$$

The Cayley-Hamilton form of the characteristic polynomial for special linear matrices provides the basic "trace" identity which allows for arbitrary free group characters to be written as polynomials of the Horowoitz generators:

$$\mathrm{tr}UV = \mathrm{tr}U\mathrm{tr}V - \mathrm{tr}UV^{-1}. \tag{2.2}$$

(Evidently the use of this identity goes back at least to Vogt [23].) For example, if $X = ABCC$, then

$$\mathrm{tr}_{ABCC} = abcc = abc \cdot c - abcc^{-1} = abc \cdot c - ab.$$

Following Culler and Shalen [1], since $\mathcal{H}_n$ generates the ring of all regular functions of the form in Equation 2.1, any free group character is determined by its evaluation on $\mathcal{H}_n$. That is, the evaluation map $T : \mathit{Hom}(F_n, SL(2,\mathbb{C})) \to \mathbb{C}^{2^n-1}$, $T(\varphi) = (\mathcal{H}_n(\varphi))$ has as its image the space of all $SL(2,\mathbb{C})$-characters of F_n. Hence

$$V_{F_n} = V_{F_{n,\mathbb{C}}} := T(\mathit{Hom}(F_n, SL(2,\mathbb{C}))$$

and $V_{F_{n,\mathbb{R}}}$ as the restriction of $V_{F_{n,\mathbb{C}}}$ to the real points.

Remark 2.2. Following Sibirskii [21], the algebra of affine invariants of a set of matrices in $SL(2,\mathbb{C})$ (as a subgroup of $GL(n,\mathbb{C})$) is minimally generated by the traces of certain products of these matrices. This minimal set is precisely the first $p = \frac{n(n^2+5)}{6}$ elements of the Horowitz Set, which we will call the Sibirskii Set. These are the Horowitz generators whose basic words have wordlength three or less in F_n. This minimal set of generators forms an integral rational basis for the algebra of invariants of n matrices in $SL(2,\mathbb{C})$, although it is not the case that all other traces of products of special linear matrices can be written as integer polynomials in the Sibirskii generators.

View these Horowitz generators as indeterminates, and the Fricke-Horowitz Theorem says that any free group character is an element of

$$\mathcal{P}_n = \mathbb{Z}\left[\mathcal{H}_n\right] = \mathbb{Z}\left[x_1, x_2, \ldots, x_{2^n-1}\right].$$

In general, however, there are polynomials with integer coefficients in these indeterminates which are identically zero for any choice of representation. These relations form an ideal $\mathcal{I}_n$ within $\mathcal{P}_n$, and the quotient ring $\mathcal{R}_n = \mathcal{P}_n/\mathcal{I}_n$ was called the *ring of Fricke characters* by Magnus [15]. In this way, the common zero locus of the elements in $\mathcal{I}_n$ forms the algebraic set V_{F_n}, which we call the *Fricke variety* of F_n.

For $n > 2$, $\mathcal{I}_n$ is not trivial. It was demonstrated by Fricke and Klein [4] that for $n = 3$, the ideal $\mathcal{I}_3$ is principal in $\mathcal{P}_3$. However, for $n > 3$, the number of generators of $\mathcal{I}_n$ grows quickly, and a full finite and minimal generating set of polynomials for $\mathcal{I}_n$ is still a topic of interest. In [5], González and Montesinos write out a complete set of generators for $\mathcal{I}_n$ under the affine embedding given by evaluation on only the Sibirskii generators. In their embedding $V_{F_n} \subset \mathbb{C}^p$, the rest of the Horowitz generators are polynomials with rational coefficients in the generating set. One can see this using a relation of Vogt [23] (see Section 6). For our embedding $V_{F_n} \subset \mathbb{C}^{2^n-1}$, we generate $\mathcal{I}_n$ by using the set constructed by Gonzalez and Montesinos relating the Sibirskii generators, and adjoining a relation via Vogt's construction for each Horowitz generator outside the Sibirskii Set. However, we will not do this explicitly here since this is not a necessary focus of this paper.

3. Polynomial automorphisms of $\mathbb{C}^{2^n-1}$

Automorphisms of F_n take bases to bases, and thus representations to representations. Hence there is an action of $Aut(F_n)$ on $Hom(F_n, SL(2,\mathbb{C}))$. The normal subgroup of inner automorphisms $Inn(F_n)$ of F_n acts by conjugation. On the (categorical) quotient space V_{F_n}, $Inn(F_n)$ acts trivially, so that there is an action of the outer automorphism group $Out(F_n) = Aut(F_n)/Inn(F_n)$ on V_{F_n}. For $\theta \in Out(F_n)$, denote by $\widehat{\theta}$ the corresponding (right) action on characters. Then (see McCool [16])

$$(\mathrm{tr}_X)\widehat{\theta} = \mathrm{tr}_{\theta(X)}. \tag{3.1}$$

By Whittemore [26], the induced $\widehat{\theta}$ is a ring automorphism of $\mathcal{R}_n$, which lifts to an automorphism of $\mathcal{P}_n$. Hence $\theta \in Out(F_n)$ induces $\widehat{\theta} \subset Aut_{\mathcal{I}_n}(\mathcal{P}_n) \subset Aut(\mathcal{P}_n)$. However, in general this lift is not unique, and one cannot extend the lift to general subgroups of $Out(F_n)$. This problem was first hinted at by Horowitz [10], and then detailed explicitly by McCool [16].

Let $F_n = \langle A_1, \ldots, A_n\rangle$. The Nielsen [19] presentation of $Aut(F_n)$ consists of inner automorphisms and four non-inner generators, given by

$$U : A_1 \mapsto A_1A_2 \quad P : \begin{array}{ccc} A_1 & \mapsto & A_2 \\ A_2 & \mapsto & A_1 \end{array} \quad Q : \begin{array}{ccc} A_i & \mapsto & A_{i+1} \\ A_n & \mapsto & A_1 \end{array} \quad \sigma : A_1 \mapsto A_1^{-1}.$$

Call these respectively twist, two-element permutation, cyclic permutation, and inversion. Effectively, these four automorphisms (really they are representatives of automorphism classes) generate $Out(F_n)$.

McCool [16] "lifts" these four Nielsen generators from their action on the quotient Fricke ring $\mathcal{R}_n$ to form an action on the ring $\mathcal{P}_n$, generated by $\widehat{U}$, $\widehat{P}$, $\widehat{Q}$ and $\widehat{\sigma}$. As the indeterminates of the Fricke ring are just the affine coordinates given by the Horowitz generators, Equation 3.1 gives a recipe for writing $\theta \in Out(F_n)$ as an integer polynomial automorphism of affine space. We will use the same notation to denote affine automorphisms, so that for $\theta \in Out(F_n)$, $\widehat{\theta} : \mathbb{C}^{2^n-1} \to \mathbb{C}^{2^n-1}$, $(\mathrm{tr}_\alpha)\widehat{\theta} = \mathrm{tr}_{\theta(\alpha)}$.

Example 3.1. Let $n = 3$, then $V_{F_3} \subset \mathbb{C}^7$ with coordinates (tr_A, tr_B, tr_C, tr_{AB}, tr_{AC}, tr_{BC}, tr_{ABC}). The coordinate polynomial of the last coordinate is

$$\begin{aligned} (\mathrm{tr}_{ABC})\,\widehat{U} &= \mathrm{tr}_{U(ABC)} = \mathrm{tr}_{ABBC} \\ &= \mathrm{tr}_{CABB} = \mathrm{tr}_{CAB}\mathrm{tr}_B - \mathrm{tr}_{CABB^{-1}} = \mathrm{tr}_{ABC}\mathrm{tr}_B - \mathrm{tr}_{AC}. \end{aligned}$$

Specific lifts of the four Nielsen generators for $n = 2$, 3, and 4, are given, respectively, in Example 6.1, Example 6.2, and Equation 5.1, below.

Define the pre-outer automorphism group $POut(F_n)$ as the group generated by $\widehat{U}$, $\widehat{P}$, $\widehat{Q}$ and $\widehat{\sigma}$. Its action on affine space has the following characteristic:

Proposition 3.2. *For any $\theta \in Out(F_n)$, the induced map $\widehat{\theta} \in POut(F_n)$ on $\mathbb{K}^{2^n-1}$ satisfies $\left|\det\left(Jac(\widehat{\theta})\right)\right| \equiv 1$.*

Proof. The above generators of $Out(F_n)$ are all of finite order except for U. In fact, $\langle P, Q, \sigma\rangle$ is the hyperoctahedral group of order n (see McCool [16]). This group has $2^n n!$ elements. Hence, any $\theta \in Out(F_n)$ in the subgroup generated by P, Q, and σ will necessarily induce a finite order polynomial automorphism $\widehat{\theta}$. Thus there exists $n \in \mathbb{N}$ where $\widehat{\theta}^n$ must be the identity, and $\left|\det\left(Jac(\widehat{\theta}^n)\right)\right| \equiv 1$. Thus the determinate of the Jacobian of $\widehat{\theta}$ is a constant root of unity. As P and σ are involutions, their induced maps must be also. Hence the proposition is satisfied for these two generators. Q is of order n. However, the induced polynomial map $\widehat{Q}$ is simply a permutation of the coordinates and hence the Jacobian of $\widehat{Q}$ will be constant and an element of $GL(2^n-1,\mathbb{Z})$. Its determinate, therefore, will either be a constant 1 or -1. This proposition will be proved once we establish the result for $\widehat{U}$. This is the content of Lemma 3.4 below. □

Remark 3.3. *A priori*, it is only understood that the polynomial maps induced by $Out(F_n)$ will be endomorphisms of affine space. However, in our case, showing the maps generated by the four Nielsen transformations are invertible is straightforward since the inverses are easily constructible and are also polynomial. In fact, it is readily apparent that $\widehat{U}$ and $\widehat{\sigma}$ are quadratic (any of the coordinate polynomials will be at most of degree 2 in the Horowitz generators), and $\widehat{Q}$ is linear (it just permutes coordinates). Again, an inspection of Equation 5.1 will help to see this. It is known, see Wang [24], that the Jacobian Conjecture holds in degrees 1 and 2: A quadratic or linear polynomial map on $\mathbb{C}^n$ with an everywhere non-vanishing Jacobian is necessarily an automorphism. Hence, except for $\widehat{P}$, showing the Jacobian of each of the generators doesn't vanish is sufficient to establish that they are automorphisms. $\widehat{P}$, however, is in general a cubic map, and the Jacobian Conjecture is still a conjecture in higher degree. Since $\widehat{P}$ is an involution, though, its square is the identity. Hence the inverse of $\widehat{P}$ is $\widehat{P}$.

Lemma 3.4. $\det\left(Jac(\widehat{U})\right)=1.$

To prove this, recall the notation where $F_n=\langle A,B,C,\ldots\rangle$ is ordered so that $A>B>C>\ldots$, and $\{x_i\}$ are the 2^n-1 Horowitz generators of P_n are ordered as stipulated in Section 2.

Lemma 3.5. *For $n>2$, the number of Horowitz generators of the form abv, where v is either the identity element or comprised of other generators, is even.*

Proof. abv corresponds to the trace of the basic word $ABV\in F_n$, where V is a basic word in the subgroup $\langle C,D,\ldots\rangle$. Thus V is also basic in F_n, and $A,B\notin V$. For $n>2$, there are $2^{n-2}-1$ such basic words V, which is odd. With $V=e$ as another choice, the number is now even. □

Recall U maps $A\mapsto AB$, and fixes all of the other generators. Hence, for two words $W,V\in F_n$, such that $A\notin W$ and $A\notin V$,

$$(wav)\widehat{U}=(\mathrm{tr}_{WAV})\widehat{U}=\mathrm{tr}_{U(WAV)}=\mathrm{tr}_{WABV}=wabv.$$

Let r_i be the ith row of $Jac(\widehat{U})$. Then $r_i=\left(\ldots,\frac{\partial}{\partial x_j}(x_i)\widehat{U},\ldots\right)$. Denote by $\lfloor r_i\rfloor=\lfloor(x_i)\widehat{U}\rfloor$ the largest index j such that $\frac{\partial}{\partial x_j}(x_i)\widehat{U}\neq 0$.

Lemma 3.6. $\forall x_i\neq av$, *where* $b\notin v$, $\lfloor(x_i)\widehat{U}\rfloor=i$.

Proof. For any word $W\in F_n$ such that $A\notin W$, $U(W)=W$. Hence $(x_i)\widehat{U}=x_i$ for any Horowitz generator where $a\notin x_i$. Thus in this case, $\lfloor(x_i)\widehat{U}\rfloor=i$. If x_i is of the form abv, for some word v, which is either basic or e, then $x_i=abv\mapsto abbv$. But $abbv=\mathrm{tr}ABBV$ and $ABBV$ is not basic. Via Equation 2.2,

$$(abv)\widehat{U}=abbv=abv\cdot b-av.$$

Note that in the image of abv under $\widehat{U}$, the lowest-ordered element is abv (shorter words are greater than longer words), so that $\lfloor (x_i)\widehat{U} \rfloor = i$. This exhausts the supply of Horowitz generators stipulated in the lemma. □

In fact, the lemma does not hold only in the cases where

$$x_i = av \overset{\widehat{U}}{\longmapsto} abv.$$

proof of Lemma 3.4 above. By Lemma 3.6, for all x_i not of the form av, $\lfloor (x_i)\widehat{U} \rfloor = i$. Thus, $Jac(\widehat{U})$ is almost lower triangular. For each v a Horowitz generator, such that $a, b \notin v$, there is a generator pair of the form $\{av, abv\}$. Under the action by $\widehat{U}$,

$$\begin{array}{rcl} (av)\widehat{U} & = & abv \\ (abv)\widehat{U} & = & abv \cdot b - av. \end{array}$$

Let e_v correspond to the elementary column operation on matrices which is the switch of the two columns corresponding to av and abv. Note after performing this switch,

$$\begin{array}{rclcl} av & \overset{\widehat{U}}{\longmapsto} & abv & \overset{e_v}{\longmapsto} & av \\ abv & \longmapsto & abv \cdot b - av & \longmapsto & av \cdot b - abv. \end{array}$$

This operation affects no other rows of $Jac(\widehat{U})$ other than the rows corresponding to av and abv. For each switch,

$$\det(e_v Jac(\widehat{U})) = -\det(Jac(\widehat{U})).$$

However, by Lemma 3.5, there are an even number of pairs. Hence after all of the switches, the determinants will be the same.

After all of these switches, the transformed Jacobian will indeed be lower triangular. Hence, its determinant can be found by simply multiplying all of the elements in the main diagonal. For all $x_i \neq abv$, the element on the main diagonal is 1. For each $x_i = abv$, the element on the main diagonal is -1. Since there are en even number of them, it follows that $\det(Jac(\widehat{U})) = 1$. □

4. Proofs of theorems

The theorems mentioned in Section 1 are now consequences of the developments of the previous sections. Here, we consolidate these developments and address the main theorems directly. We start with Theorem 1.1:

Theorem 1.1. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $POut(F_n)$ acts as volume-preserving, polynomial automorphisms on $\mathbb{K}^{2^n-1}$ and restricts to the action of $Out(F_n)$ on $V_{F_n,\mathbb{K}} \subset \mathbb{K}^{2^n-1}$.*

Proof. Any $\widehat{\theta} \in POut(F_n)$ is a composition of some combination of the four Nielsen generators above and hence will act as a polynomial automorphism of $\mathbb{K}^{2^n-1}$. By Proposition 3.2, $\left|\det\left(Jac(\widehat{\theta})\right)\right| = 1$. Let ν be the analytic 2^n-1 form on $\mathbb{K}^{2^n-1}$

$$\nu = \bigwedge_{i=1}^{2^n-1} dx_i \in \bigwedge{}^{2^n-1}\mathbb{K}^{2^n-1}$$

formed by the differentials of the Horowitz generators. Then

$$\nu \longmapsto \det\left(Jac(\widehat{\theta})\right) \cdot \nu.$$

Thus, up to sign, ν is invariant under $\widehat{\sigma}$. Hence the volume element $|\nu|$ is invariant under $\widehat{\theta}$. $\square$

Theorem 1.2. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $PMCG(S)$ acts as constant unit Jacobian polynomial automorphisms on $\mathbb{K}^{2^n-1}$ and restricts to the Poisson action of $MCG(S)$ on $V_{S,\mathbb{K}} \subset \mathbb{K}^{2^n-1}$.*

If we restrict to the real points, then $V_{F_n,\mathbb{R}} = V_{F_n} \cap \mathbb{R}^{2^n-1}$ is a real algebraic set. The $SL(2,\mathbb{C})$-characters of F_n which are real-valued come from one of the real forms of $SL(2,\mathbb{C})$. Morgan and Shalen [18] show that the real forms of $SL(2,\mathbb{C})$ come in two types: $SL(2,\mathbb{R})$ and $SU(2)$. Let $F_n = \pi_1(S)$ for some compact orientable surface S. Then S is of genus-g with $k = n - 2g + 1 > 0$ boundary components. Here V_{F_n} has the additional structure of a Poisson variety, whose symplectic leaves are the inverse images of points of the Casimir map: Let $\{C_i\}_{i=1}^k \subset F_n = \pi_1(S)$ be the set of simple loops homotopic to each of the boundary components of S. Then the map

$$F : Hom(S, SL(2,\mathbb{C})) \longrightarrow \mathbb{C}^k$$

where $F(\phi) = (tr_{C_1}(\phi), \ldots, tr_{C_k}(\phi)) = (c_1, \ldots, c_k)$ is conjugation invariant and descends to a map on V_{F_n}. The Poisson structure defines a symplectic structure on these leaves. It is known that the mapping class group of the surface $MCG(S) \subset Out(\pi_1(S))$ preserves this Poisson structure, and acts symplectically on each of the leaves. Hence, the symplectic volume ν_s on each of these leaves is preserved under the action of mapping classes. One can define a volume form ν_ℓ from the leaf space on which the mapping class group acts trivially (it preserves the leaf structure), Then, it is easy to see that for $\theta \in MCG(S)$, $\widehat{\theta}$ preserves the volume form on V_{F_n} given by $\nu_s \wedge F^*\nu_\ell$.

To prove the theorem, for $\theta \in Out(\pi_1(S))$, let $\theta_H \in Aut(H_1(S;\mathbb{Z}))$ be its associated linear action on first homology.

Lemma 4.3. *For $\theta \in MCG(S)$, θ_H is unimodular.*

For any particular surface S with F_n as its fundamental group, a presentation of $MCG(S)$ would lead to a direct calculation of the corresponding action on the abelianization of F_n, which is $H_1(S)$. Instead, to prove this assertion without regard to any particular choice of S, we will appeal via Poincaré duality to the effect of a mapping class on first cohomology. We thank Bill Goldman for this suggestion, and a discussion of the proof.

Proof. To start, let $\partial S = \emptyset$ (while $\pi_1(S)$ would not be free in this case, this case is instructive). $H_1(S)$ is Poincaré dual to $H^1(S)$. There is a nondegenerate, skew-symmetric, bilinear 2-form on $H^1(S)$ given by cup product.

$$\omega : H^1(S) \times H^1(S) \longrightarrow H^2(S) \cong \mathbb{Z}.$$

This structure forms the basis for the construction of a symplectic structure on the G-character variety of S, for G a Lie group with an orthogonal structure (see Goldman [6]). It is known that the induced map θ^* preserves this cup product, and acts symplectically on the G-character variety of S. Hence the linear action on $H^1(S)$ and thus on $H_1(S)$ is by symplectic matrix, which is unimodular.

For our case, where necessarily $\partial S \neq \emptyset$, $H^2(S) = 0$, the above construction fails. There is a corresponding relative version of the cup product, given by

$$\omega : H^1(S, \partial S) \times H^1(S, \partial S) \longrightarrow H^2(S, \partial S) \cong \mathbb{Z},$$

which is degenerate. However, by restricting this product to the parabolic cohomology

$$\omega_p : H^1_p(S, \partial S) \times H^1_p(S, \partial S) \longrightarrow H^2_p(S, \partial S) \cong H^2(S, \partial S),$$

the 2-form is nondegenerate. Note that parabolic cocycles relative to ∂S are relative cocycles of S which restrict to coboundaries on ∂S (see Weil [25]). This is essentially the construction of Huebschmann [11] to establish the symplectic structure on the symplectic leaves of the Poisson character variety of a surface with boundary mentioned above. θ^* preserves this cup product, and leaves invariant the short exact sequence

$$0 \longrightarrow H^1_p(S, \partial S) \longrightarrow H^1(S) \longrightarrow H^1(\partial S) \longrightarrow 0.$$

By definition, mapping classes fix pointwise the components of ∂S. Hence, the determinant of θ^* on $H^1(S)$ must equal the product of the determinant of θ^* on $H^1_p(S, \partial S)$ and that of θ^* on $H^1(\partial S)$. As mapping classes of the surface act identically on the boundary cohomology, θ^* on $H^1(S)$ must be unimodular. □

Lemma 4.4. *For $\theta \in Out(F_n)$,*

$$\det(\theta_H) = \det(Jac(\widehat{\theta})).$$

The proof of this Lemma can be established via a direct calculation, given the four Nielsen generators of $Out(F_n)$ and their associated actions of the relevant spaces. We leave this for the reader.

Proof of Theorem 1.2. The theorem will now follow as a corollary of the above lemmas. Since outer automorphisms of $F_n = \pi_1(S)$ which correspond to mapping classes (orientation preserving isotopy classes of homeomorphisms) of S lead to unimodular automorphisms of $H_1(S)$, they will act as polynomial automorphisms of $\mathbb{R}^{2^n-1}$ whose Jacobian is everywhere 1. By the same argument given in Theorem 1.1 above, an automorphism of $\mathbb{R}^{2^n-1}$ will take the standard real volume form ν to a functional multiple of itself, so that

$$\nu \longmapsto \det\left(Jac(\widehat{\theta})\right) \cdot \nu.$$

Since for $\theta \in MCG(S)$, we have $\det\left(Jac(\widehat{\theta})\right) = 1$ everywhere, it follows that $\widehat{\theta}$ acts as a volume preserving automorphism of $\mathbb{R}^{2^n-1}$. □

5. The kernel of the McCool epimorphism

In [16], McCool shows that for $n < 4$, the kernel of the map $\Phi_n : POut(F_n) \to Out(F_n)$ is trivial. For $n \geq 4$, he constructs an example of a non-trivial element in $\ker \Phi_n$ (see Example 6.3 for the rank-4 version of this example). Here, we will analyze the structure of $\ker \Phi_n$, and show that $\ker \Phi_4$ is the universal, rank-3 Coxeter group. We will also find a set of three generators for $\ker \Phi_n$, $n > 4$, and show that the generators are of infinite order.

To start, return to the generating set for $Out(F_n)$ given by Nielsen from Section 3,

$$U: A_1 \mapsto A_1A_2 \quad P: \begin{array}{ccc} A_1 & \mapsto & A_2 \\ A_2 & \mapsto & A_1 \end{array} \quad Q: \begin{array}{ccc} A_i & \mapsto & A_{i+1} \\ A_n & \mapsto & A_1 \end{array} \quad \sigma: A_1 \mapsto A_1^{-1}.$$

Nielsen's presentation of $Aut(F_n)$ included inner automorphisms and these four generators. And Nielsen's relations for $Aut(F_n)$ included the following 18 non-inner relations (some of these are a slight variation of the originals):

$$P^2 = \sigma^2 = (PQ)^3 = Q^4 = 1,$$

$$[Q^2PQ^{-2}, P] = [\sigma, Q^{-1}PQ] = [Q^{-1}P, \sigma] = [Q\sigma Q^{-1}, \sigma] = 1,$$

$$(UP\sigma P)^2 = \sigma P\sigma U\sigma PUPU^{-1} = U^{-1}QU^{-1}Q^{-1}U\left(QUQ^{-1}P\right)^2 = 1,$$

$$[Q^2PQ^{-2}, U] = [Q^{-1}PQPQ^{-1}, U^{-1}] = [Q^2\sigma Q^{-2}, U] = [Q^2UQ^{-2}, U] = 1,$$

$$[\sigma U\sigma, U] = [PQ\sigma U^{-1}\sigma Q^{-1}P, U] = [PQPQ^{-1}PU^{-1}PQPQ^{-1}P, U] = 1.$$

Designate the four lifts $\widehat{U}$, $\widehat{P}$, $\widehat{\sigma}$, and $\widehat{Q}$, following the notation of McCool. By direct calculation, all of these relators lift to the identity polynomial automorphism of $\mathbb{K}^{2^n-1}$ except for the three

$$R_1 = [Q^{-1}PQPQ^{-1}, U^{-1}], \quad R_2 = U^{-1}QU^{-1}Q^{-1}U\left(QUQ^{-1}P\right)^2 = 1, \quad R_3 = (PQ)^{n-1}.$$

McCool [17] showed that for $n > 4$, the automorphism $\widehat{R}_3$ is of infinite order. Herein, we will show that for $n = 4$, $\widehat{R}_3$ is of order 2 and for $n \geq 4$, $\widehat{R}_1$ and $\widehat{R}_2$ are of infinite order. Together these three polynomial maps generate $\ker \Phi_n$, although for $n = 4$ there is a preferred generating set.

5.1. **Finite generators: The rank-4 case.** Present $F_4 = \langle A, B, C, D\rangle$ and order the generators so that $A > B > C > D$. Then the ordered Horowitz generating set is given by the 15 characters

$$\begin{aligned} \{x_i\}_{i=1}^{15} &= \{a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\} \\ &= \{l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}. \end{aligned}$$

Using these characters as affine coordinates of $\mathbb{K}^{15}$, the McCool generators take the form

(5.1)

$$\widehat{U}: \begin{array}{ccl} l & \mapsto & p \\ m & \mapsto & m \\ n & \mapsto & n \\ o & \mapsto & o \\ p & \mapsto & mp - l \\ q & \mapsto & v \\ r & \mapsto & w \\ s & \mapsto & s \\ t & \mapsto & t \\ u & \mapsto & u \\ v & \mapsto & mv - q \\ w & \mapsto & mw - r \\ x & \mapsto & z \\ y & \mapsto & y \\ z & \mapsto & mz - x \end{array} \quad \widehat{P}: \begin{array}{ccl} l & \mapsto & m \\ m & \mapsto & l \\ n & \mapsto & n \\ o & \mapsto & o \\ p & \mapsto & p \\ q & \mapsto & s \\ r & \mapsto & t \\ s & \mapsto & q \\ t & \mapsto & r \\ u & \mapsto & u \\ v & \mapsto & -lmn + ls + mq + np - v \\ w & \mapsto & -lmo + lt + mr + op - w \\ x & \mapsto & y \\ y & \mapsto & x \\ z & \mapsto & -lmu + ly + mx + pu - z \end{array} \quad \widehat{Q}: \begin{array}{ccl} l & \mapsto & m \\ m & \mapsto & n \\ n & \mapsto & o \\ o & \mapsto & l \\ p & \mapsto & s \\ q & \mapsto & t \\ r & \mapsto & p \\ s & \mapsto & u \\ t & \mapsto & q \\ u & \mapsto & r \\ v & \mapsto & y \\ w & \mapsto & v \\ x & \mapsto & w \\ y & \mapsto & x \\ z & \mapsto & z \end{array} \quad \widehat{\sigma}: \begin{array}{ccl} l & \mapsto & l \\ m & \mapsto & m \\ n & \mapsto & n \\ o & \mapsto & o \\ p & \mapsto & lm - p \\ q & \mapsto & ln - q \\ r & \mapsto & lo - r \\ s & \mapsto & s \\ t & \mapsto & t \\ u & \mapsto & u \\ v & \mapsto & ls - v \\ w & \mapsto & lt - w \\ x & \mapsto & lu - x \\ y & \mapsto & y \\ z & \mapsto & ly - z \end{array}$$

Now instead of using the three nontrivial relations $\widehat{R}_i$, $i = 1, 2, 3$ directly, consider the relations $\widehat{K}_3 = \widehat{R}_3$, and for $i = 1, 2$, $\widehat{K}_i = \widehat{R}_i\widehat{R}_3$. For $n = 4$, these three new relations lift to the polynomial maps on $\mathbb{K}^{15}$

$$
\widehat{K}_1 : \begin{array}{rcl} l & \mapsto & l \\ m & \mapsto & m \\ n & \mapsto & n \\ o & \mapsto & o \\ p & \mapsto & p \\ q & \mapsto & q \\ r & \mapsto & r \\ s & \mapsto & s \\ t & \mapsto & t \\ u & \mapsto & u \\ v & \mapsto & v \\ w & \mapsto & -lmo - lny + lsu \\ & & +lt + mr + nz + op \\ & & +qy - sx - uv - w \\ x & \mapsto & x \\ y & \mapsto & y \\ z & \mapsto & z \end{array}
\quad
\widehat{K}_2 : \begin{array}{rcl} l & \mapsto & l \\ m & \mapsto & m \\ n & \mapsto & n \\ o & \mapsto & o \\ p & \mapsto & p \\ q & \mapsto & q \\ r & \mapsto & r \\ s & \mapsto & s \\ t & \mapsto & t \\ u & \mapsto & u \\ v & \mapsto & v \\ w & \mapsto & w \\ x & \mapsto & -lno - mov + ops \\ & & +lu + mz + nr + oq \\ & & -py - sw + tv - x \\ y & \mapsto & y \\ z & \mapsto & z \end{array}
\quad
\widehat{K}_3 : \begin{array}{rcl} l & \mapsto & l \\ m & \mapsto & m \\ n & \mapsto & n \\ o & \mapsto & o \\ p & \mapsto & p \\ q & \mapsto & q \\ r & \mapsto & r \\ s & \mapsto & s \\ t & \mapsto & t \\ u & \mapsto & u \\ v & \mapsto & v \\ w & \mapsto & w \\ x & \mapsto & y \\ y & \mapsto & y \\ z & \mapsto & lmno - lmu - los \\ & & -mnr - nop + ly \\ & & +mx + nw + ov \\ & & +pu + rs - qt - z \end{array}
\tag{5.2}
$$

It is interesting to note that $(z)\widehat{K}_3 = (abcd)\widehat{K}_3$ above is directly related to the Vogt Relation, a member of $\mathcal{I}_4$ (indeed, a member of $\mathcal{I}_n$ for all $n > 3$). See Vogt [23] or Example 6.3. Recall a Coxeter group is a group generated by involutions s_i, whose relations have the form $(s_i s_j)^{k_{ij}}$ for $k_{ij} \in \mathbb{N} \cup \infty$ (Note $k_{ii} = 1$). When the exponents $k_{ij} = \infty$, $\forall i \neq j$, then there are no relations beyond the involutions themselves, and the Coxeter group is called universal. In our case, it is a quick calculation to see that $\widehat{K}_i^2$ is the identity for each $i = 1, 2, 3$. What is less obvious is that these are the only relations between these three polynomial maps.

Theorem 1.3. $\ker \Phi_4$ *is the universal, rank-3 Coxeter group.*

To prove this, restrict these automorphisms to the 3-dimensional hyperplane given by the family of vectors:

$$\left\{ \mathbf{u} = (0, 2, 2, 0, 0, 0, 0, -2, 0, 0, 0, w, x, 0, z) \in \mathbb{K}^{15} \,\middle|\, w, x, z \in \mathbb{K} \right\}.$$

Call this hyperplane V. V is a vector space parameterized by x, y, z, and whose elements are of the form $\mathbf{v} = \mathbf{u} - \mathbf{u}_0$, where $\mathbf{u}_0 \in V$ has $w = x = z = 0$. For $i = 1, 2, 3$, V is invariant under $\widehat{K}_i$, and restricted to V we have

$$
\widehat{K}_1\Big|_V : \begin{array}{rcl} w & \mapsto & 2(x+z) - w \\ x & \mapsto & x \\ z & \mapsto & z \end{array}
\quad
\widehat{K}_2\Big|_V : \begin{array}{rcl} w & \mapsto & w \\ x & \mapsto & 2(w+z) - x \\ z & \mapsto & z \end{array}
\quad
\widehat{K}_3\Big|_V : \begin{array}{rcl} w & \mapsto & w \\ x & \mapsto & x \\ z & \mapsto & 2(w+x) - z, \end{array}
$$

all of which are linear.

Now let $\| \cdot \|$ be the max norm on V, so that for $\mathbf{v} \in V$, $\|\mathbf{v}\| = \max_i \{|\mathbf{v}(i)|\}$. Then, for $\mathbf{v}_0 = (1, 1, 1)$, $\|\mathbf{v}_0\| = 1$ and $\|(\mathbf{v}_0)\widehat{K}_i\| = 3$. Given a reduced word $\overline{w} = w_1 w_2 \cdots w_n \in \ker \Phi_4$, where for each $w_j = \widehat{K}_i$, for $i \in \{1, 2, 3\}$, denote by $\mathbf{v}_j = (\mathbf{v}_{j-1})w_j = (\mathbf{v}_0)w_1 \cdots w_{j-1}w_j$. We have the following:

Proposition 5.1. *Given* $\overline{w} = w_1 w_2 \cdots w_n$*, and* $\mathbf{v}_0 = (1, 1, 1)$*, where* $w_n = \widehat{K}_i$*,* $i \in \{1, 2, 3\}$*, we have* $\|\mathbf{v}_n\| = \|(\mathbf{v}_0)\overline{w}\| = \mathbf{v}_n(i) \in \mathbb{Z}$ *and all entries of* $\mathbf{v}_n$ *are positive.*

Proof. We will use the three subscripts $i \neq j \neq k$ to keep the result general. Note the claim is certainly true for $n = 1$, as $(\mathbf{v}_0)\widehat{K}_i$ has a 3 in the ith position, and 1's in the other two spots. Inductively, suppose the claim holds up to the $(n-1)$st stage, so that $\|\mathbf{v}_{n-1}\| = \|(\mathbf{v}_{n-2})\widehat{K}_j\| = \mathbf{v}_{n-1}(j) > 1$ (again $j \neq i$), and all entries of $\mathbf{v}_{n-1}$ are positive.

Now $\mathbf{v}_n = (\mathbf{v}_{n-1})\widehat{K}_i$ has the entries

$$\begin{aligned}\mathbf{v}_n(i) &= 2\left(\mathbf{v}_{n-1}(j) + \mathbf{v}_{n-1}(k)\right) - \mathbf{v}_{n-1}(i)\\ \mathbf{v}_n(j) &= \mathbf{v}_{n-1}(j)\\ \mathbf{v}_n(k) &= \mathbf{v}_{n-1}(k).\end{aligned}$$

Combining these, we get

$$\mathbf{v}_n(i) = 2\left(\mathbf{v}_n(j) + \mathbf{v}_n(k)\right) - \mathbf{v}_{n-1}(i).$$

And since $\|\mathbf{v}_{n-1}\| = \mathbf{v}_{n-1}(j)$ by assumption, we have

$$\mathbf{v}_n(j) = \mathbf{v}_{n-1}(j) > \mathbf{v}_{n-1}(i).$$

Hence

$$\begin{aligned}\mathbf{v}_n(i) &= 2\left(\mathbf{v}_n(j) + \mathbf{v}_n(k)\right) - \mathbf{v}_{n-1}(i)\\ &> 2\left(\mathbf{v}_n(j) + \mathbf{v}_n(k)\right) - \mathbf{v}_n(j) = \mathbf{v}_n(j) + 2\mathbf{v}_n(k).\end{aligned}$$

Hence $\mathbf{v}_n(i) > \max\{\mathbf{v}_n(j), \mathbf{v}_n(k)\}$, and $\|\mathbf{v}_n\| = \mathbf{v}_n(i)$. Since $\mathbf{v}_n(j) = \mathbf{v}_{n-1}(j) > 0$ and $\mathbf{v}_n(k) = \mathbf{v}_{n-1}(k) > 0$, we are done. □

We have some immediate corollaries. Denote the word length of $\overline{w}$ by $\ell(\overline{w})$:

Corollary 5.2. *Let $\overline{w} \in \ker \Phi_4$. Then for $\mathbf{v}_0 = (1,1,1)$, $\|(\mathbf{v}_0)\overline{w}\| > \ell(\overline{w})$.*

Proof. For any choice of $i \in \{1,2,3\}$, $\|\mathbf{v}_1\| = \|(\mathbf{v}_0)\widehat{K}_i\| = 3$. By Proposition 5.1, for any $(\mathbf{v}_0)\overline{w}$, the entries are positive integers, and

$$\|\mathbf{v}_n\| = \|(\mathbf{v}_{n-1})\widehat{K}_i\| \geq \|\mathbf{v}_{n-1}\| + 1.$$

□

Also the following corollary now follows immediately.

Corollary 5.3. *There does not exist $\overline{w} \in \ker \Phi_4$, where $\ell(\overline{w}) > 1$, such that $\overline{w} = 1$.*

Proof. If there were such a $\overline{w}$, then $(\mathbf{v}_0)\overline{w} = \mathbf{v}_0$ and $\|(\mathbf{v}_0)\overline{w}\| = \|\mathbf{v}_0\|$. But this is not possible, as by Corollary 5.2,

$$\|(\mathbf{v}_0)\overline{w}\| > \ell(\overline{w}) > 1 = \|\mathbf{v}_0\|.$$

□

Hence, besides the squares of the three generators, there are no other relations, and $\ker \Phi_4 = \langle \widehat{K}_1, \widehat{K}_2, \widehat{K}_3 \,\big|\, \widehat{K}_1^2 = \widehat{K}_2^2 = \widehat{K}_3^2 \rangle$. Theorem 1.3 is now established.

Lastly, the restrictions of the generators K_i of $\ker \Phi_4$ to V is a representation

$$\varphi\big|_V : \ker \Phi_4 \longrightarrow GL(V), \text{ where } K_i \mapsto K_i\big|_V.$$

Corollary 5.4. *$\varphi\big|_V$ is faithful.*

Proof. This will be established if we can show that for any $\overline{w} \neq \overline{u} \in \ker \Phi_4$ and $\mathbf{v}_0 = (1,1,1)$, we have $(\mathbf{v}_0)\overline{w} \neq (\mathbf{v}_0)\overline{u}$. to this end, suppose $\overline{w} \neq \overline{u}$. Then either $\ell(\overline{w}) \neq \ell(\overline{u})$, or $\ell(\overline{w}) = \ell(\overline{u}) > 1$, and $\exists i \in \{1, \ldots, \ell(\overline{w})\}$, where $w_i \neq u_i$. Suppose that $(\mathbf{v}_0)\overline{w} = (\mathbf{v}_0)\overline{u}$. Then $(\mathbf{v}_0)\overline{u}\overline{w}^{-1} = \mathbf{v}_0$. But by Corollary 5.2, this is impossible, since

$$\|(\mathbf{v}_0)\overline{u}\overline{w}^{-1}\| > \ell(\overline{u}\overline{w}^{-1}) > 1 = \|\mathbf{v}_0\|.$$

Thus for any two distinct $\overline{w}$ and $\overline{u}$, we have $(\mathbf{v}_0)\overline{w} \neq (\mathbf{v}_0)\overline{u}$. □

Remark 5.5. The representation $\varphi\big|_V$ given in Corollary 5.4 is precisely the geometric representation associated to any Coxeter system (See Humphries [12], or Qi [20]). For a rank-n Coxeter system W, one associates a representation on an n-vector space via the construction of a $n \times n$ symmetric matrix M_W(called the Coxeter matrix), whose i,jth entry is the order of the cyclic subgroup generated by the product of the pair of generators K_iK_j, where $i,j \in \{1,2,3\}$. Since Coxeter groups are generated by involutions, the main diagonal of this matrix consists of 1's. M_W is also symmetric since $\#(K_jK_i) = \#(K_iK_j)$, and by convention, when $\#(K_iK_j) = \infty$ for $i \neq j$, the i,jth entry of M_W is given a -1. The universal Coxeter group in rank 3 has

$$M_W = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix},$$

and given the standard basis $\mathbf{x}_i$, with a 1 in the ith spot and zeros otherwise, we have

$$(\mathbf{v})K_i = \mathbf{v} - 2\left(\mathbf{x}_i^T M_W \mathbf{v}\right)\mathbf{x}_i.$$

The fact that we have unearthed this representation from the polynomial automorphisms themselves is an interesting side effect of this analysis.

5.2. **Invariant subspaces and ideals: Higher rank cases.** McCool's [17] technique for showing that for $n > 4$, the automorphism $\widehat{R}_3$ is of infinite order involved a recognition that the ideal of functions generated by the Horowitz generators of word-length 1 (the characters of the individual generators of F_n) is invariant under the action of $\widehat{R}_3$. He then constructs an infinite orbit of the map modulo this invariant ideal. This act of simplifying the action to better "see" the orbits is similar to one we will use presently. In this section, we will show that the other two generators of $\ker \Phi_n$, $n \geq 4$ have the same property. This will establish Theorem 1.4.

For $n > 3$, $F_n = \langle A, B, C, D, E, \ldots, L\rangle$, use a slight variation of the notation in the proof of Lemma 3.5: Let v be the Horowitz generator of a basic word that does not include the letters A, B or C. Call $v_m = \mathrm{tr} DE \cdots L$ the character of the unique longest basic word of this type. Consider the subspaces of $\mathbb{K}^{2^n-1}$ given by

$$V_1 = \mathrm{span}\left\{c, abv_m, abcv_m\right\}, \text{ and } V_2 = \left\{b, acv_m, abcv_m\right\},$$

and denote, respectively, the variables $\{x, y, z\}$ a coordinate system for each of the V_i. We have the following:

Proposition 5.6. *For $i = 1, 2$, V_i is invariant under $\widehat{R}_i$, and*

$$\left(\widehat{R}_i\Big|_{V_i}\right)^n : \begin{array}{ccl} x & \mapsto & x \\ y & \mapsto & \mathcal{F}_{2n-2}(x)y + \mathcal{F}_{2n-1}(x)z \\ z & \mapsto & \mathcal{F}_{2n-1}(x)y + \mathcal{F}_{2n}(x)z, \end{array}$$

where $\mathcal{F}_n(x) = x\mathcal{F}_{n-1}(x) - \mathcal{F}_{n-2}(x)$ is the nth alternating Fibonacci polynomial (of the second kind) with initial data $\mathcal{F}_0(x) = 1$ and $\mathcal{F}_1(x) = x$.

Proof. $\widehat{R_1}$ fixes every Horowitz generator of word length less than three. It also fixes all Horowitz generators that do not contain the letter a. And it fixes the generator abc. And on the remaining generators, we have

$$\begin{array}{rrcl} & abv & \mapsto & -abv+av\cdot b-acv\cdot bc+ac\cdot bcv+a\cdot bv+abcv\cdot c-a\cdot bcv\cdot c \\ & & & -abc\cdot cv+a\cdot bc\cdot cv+ab\cdot v-a\cdot b\cdot v \\ \widehat{R_1}: & abcv & \mapsto & -abcv+acv\cdot b+av\cdot bc+a\cdot bcv-ac\cdot bv-abv\cdot c-acv\cdot bc\cdot c \\ & & & +ac\cdot bcv\cdot c+a\cdot bv\cdot c+abcv\cdot c^2-a\cdot bcv\cdot c^2+ab\cdot cv-a\cdot b\cdot cv \\ & & & -abc\cdot c\cdot cv+a\cdot bc\cdot c\cdot cv+abc\cdot v-a\cdot bc\cdot v. \end{array}$$

If we choose a v as above, and consider only initial vectors with non-zero values in the subspace generated by the span of c, abv, and $abcv$, we see that this subspace is invariant under $\widehat{R}_1$, and

$$\begin{array}{rrcl} & c & \mapsto & c \\ \widehat{R_1}: & abv & \mapsto & -abv+abcv\cdot c \\ & abcv & \mapsto & -abcv-abv\cdot c+abcv\cdot c^2. \end{array}$$

Let $v=v_m$, and re-parameterize $x=c$, $y=abv_m$ and $z=abcv_m$. Then

$$\begin{array}{rrcl} & x & \mapsto & x \\ \widehat{R_1}: & y & \mapsto & -y+xz \\ & z & \mapsto & -z-xy+x^2z. \end{array}$$

This automorphism generates a coupled system of two first order recursions in y and z. Rewritten as a single vector recursion, we get

$$\begin{aligned} \begin{bmatrix} y \\ z \end{bmatrix}_{n+1} &= \begin{bmatrix} -1 & x \\ -x & x^2-1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}_n \\ &= \begin{bmatrix} 0 & -1 \\ 1 & -x \end{bmatrix}^2 \begin{bmatrix} y \\ z \end{bmatrix}_n. \end{aligned}$$

The matrix $\begin{bmatrix} 0 & -1 \\ 1 & -x \end{bmatrix}$ is the generating matrix of an alternating Fibonacci polynomial sequence of the second sind (See Kosby [14] and Kapproff [13]). The result follows. Note the fact that the generating matrix is squared is reflected in the subscripts of the result.

The $\widehat{R_2}$ case is similar. As before, $\widehat{R}_2$ fixes every Horowitz generator of word length less than three, as well as any generator that does not begin with a. It also fixes all generators that do not end in the last letter ℓ. It also fixes all words of the form $ab\ell$. On the remaining generators, let v be the character of a basic word in $\langle C, D, \ldots, L\rangle$ that does not include L. Then we have

$$\begin{array}{rrcl} & av\ell & \mapsto & -av\ell+abv\ell\cdot b-ab\ell\cdot bv-ab\cdot bv\ell+abv\cdot b\ell+a\ell\cdot v+a\cdot v\ell \\ & & & +av\cdot\ell-abv\cdot b\ell+ab\cdot bv\cdot\ell-a\cdot v\cdot\ell \\ \widehat{R_2}: & abv\ell & \mapsto & -abv\ell-av\ell\cdot b+abv\ell\cdot b^2+a\ell\cdot bv-ab\ell\cdot bbv+a\cdot bv\ell-ab\cdot b\cdot bv\ell \\ & & & -av\cdot b\ell+abv\cdot b\cdot b\ell+ab\ell\cdot v+ab\cdot v\ell+abv\cdot\ell+av\cdot b\cdot\ell-abv\cdot b^2\cdot\ell \\ & & & -a\cdot bv\cdot\ell+ab\cdot b\cdot bv\cdot\ell-ab\cdot v\cdot\ell. \end{array}$$

The subspace generated by the span of b, $av\ell$, and $abv\ell$ is invariant under $\widehat{R}_2$. Restricted to an initial vector in this subspace, we get

$$\begin{array}{rrcl} & b & \mapsto & b \\ \widehat{R_2}: & av\ell & \mapsto & -av\ell+abv\ell\cdot b \\ & abv\ell & \mapsto & -abv\ell-av\ell\cdot b+abv\ell\cdot b^2. \end{array}$$

Let v correspond to the basic word of longest length in $\langle D, E, \ldots, L\rangle$. Then $v_m = v$. Again, re-parameterize $x = b$, $y = acv_m$ and $z = abcv_m$, and

$$\widehat{R_2}: \begin{array}{ccl} x & \mapsto & x \\ y & \mapsto & -y + xz \\ z & \mapsto & -z - xy + x^2 z \end{array}$$

Exactly as before. This completes the proof. □

And with the previously mentioned result by McCool, we have the following.

Theorem 5.7. *For $n > 4$, $\ker \Phi_n$ is generated by three non-torsion elements.*

6. Examples

Example 6.1. Let $F_2 = \langle A, B\rangle$, ordered so that $A > B$. The Horowitz generating set is given by the three characters a, b, and ab. It was proven by Fricke [4] that the character ring of F_2 is freely generated by these three functions, and $\mathcal{I}_2$ is trivial. Hence $V_{F_2} \equiv \mathbb{K}^3$ (compare also Horowitz [9] and Whittemore [26]). $Out(F_2)$ acts as polynomial automorphisms of $\mathbb{K}^3$ via the four (actually three) generators

$$\widehat{U}: \begin{array}{ccl} a & \mapsto & ab \\ b & \mapsto & b \\ ab & \mapsto & b \cdot ab - a \end{array} \qquad \widehat{P}, \widehat{Q}: \begin{array}{ccl} a & \mapsto & b \\ b & \mapsto & a \\ ab & \mapsto & ab \end{array} \qquad \widehat{\sigma}: \begin{array}{ccl} a & \mapsto & a \\ b & \mapsto & b \\ ab & \mapsto & a \cdot b - ab \end{array}$$

with infinitesimal data

$$Jac\left(\widehat{U}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & ab & b \end{bmatrix}, \quad Jac\left(\widehat{P}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Jac\left(\widehat{\sigma}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & a & -1 \end{bmatrix}.$$

Example 6.2. For $F_3 = \langle A, B, C\rangle$, where $A > B > C$, the Fricke character ring is the quotient of

$$\mathcal{P}_3 = \mathbb{Z}[a, b, c, ab, ac, bc, abc]$$

by the single generator of $\mathcal{I}_3$ given by the Fricke Relation:

$$\begin{array}{rcl} p_1 & = & a \cdot b \cdot c \cdot abc - a \cdot b \cdot ab - a \cdot c \cdot ac + a \cdot bc \cdot abc - b \cdot c \cdot bc + b \cdot ac \cdot abc \\ & & +c \cdot ab \cdot abc + ab \cdot ac \cdot bc + a^2 + b^2 + c^2 + ab^2 + ac^2 + bc^2 + abc^2 - 4. \end{array}$$

Hence $V_{F_n} \subset \mathbb{K}^7$ is the zero locus of p_1. In this case, The action of $Out(F_n)$ is via the four generators

$$\widehat{U}: \begin{array}{ccl} a & \mapsto & ab \\ b & \mapsto & b \\ c & \mapsto & c \\ ab & \mapsto & b \cdot ab - a \\ ac & \mapsto & abc \\ bc & \mapsto & bc \\ abc & \mapsto & b \cdot abc - ac \end{array} \qquad \widehat{P}: \begin{array}{ccl} a & \mapsto & b \\ b & \mapsto & a \\ c & \mapsto & c \\ ab & \mapsto & ab \\ ac & \mapsto & bc \\ bc & \mapsto & ac \\ abc & \mapsto & -a \cdot b \cdot c + a \cdot bc \\ & & +b \cdot ac + c \cdot ab - abc \end{array}$$

$$\widehat{Q}: \begin{array}{ccl} a & \mapsto & b \\ b & \mapsto & c \\ c & \mapsto & a \\ ab & \mapsto & bc \\ ac & \mapsto & ab \\ bc & \mapsto & ac \\ abc & \mapsto & abc \end{array} \qquad \widehat{\sigma}: \begin{array}{ccl} a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & c \\ ab & \mapsto & a \cdot b - ab \\ ac & \mapsto & a \cdot c - ac \\ bc & \mapsto & bc \\ abc & \mapsto & a \cdot bc - abc \end{array}$$

As in the $n = 2$ case, $Out(F_3) \cong POut(F_3)$ (See McCool [16]), so we may speak of the $Out(F_3)$-action on the ambient affine space $\mathbb{K}^7$.

Example 6.3. As in Section 5.1, let $F_4 = \langle A, B, C, D\rangle$, ordered so that $A > B > C > D$. Then the Horowitz generating set is

$$\begin{array}{rcl} \{x_i\}_{i=1}^{15} & = & \{a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\} \\ & = & \{l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}, \end{array}$$

In Equation 5.2, we detailed the automorphism $\widehat{K}_3 = \left(\widehat{P}\widehat{Q}\right)^3$, noting that the composition PQ was of order 3 in $Out(F_4)$, but

$$\left(\widehat{P}\widehat{Q}\right)^3 : \begin{array}{rcl} l & \mapsto & l \\ m & \mapsto & m \\ n & \mapsto & n \\ o & \mapsto & o \\ p & \mapsto & p \\ q & \mapsto & q \\ r & \mapsto & r \\ s & \mapsto & s \\ t & \mapsto & t \\ u & \mapsto & u \\ v & \mapsto & v \\ w & \mapsto & w \\ x & \mapsto & x \\ y & \mapsto & y \\ z & \mapsto & lmno - lmu - los - mnr - nop + ly + mx + nw + ov + pu - qt + rs - z \end{array}$$

is not the identity map on $\mathbb{K}^{15}$. On V_{F_n}, the last coordinate polynomial is precisely z. This suggests that the polynomial

$$lmno - lmu - los - mnr - nop + ly + mx + nw + ov + pu - qt + rs - 2z \in \mathcal{I}_4. \tag{6.1}$$

This relation was well known to Vogt [23], and does not vanish off of V_{F_n}. As we have seen, $\widehat{K}_3$ is an involution on $\mathbb{K}^{15}$, so that $(\widehat{P}\widehat{Q})$ is of order 6 in $POut(F_4)$. This is precisely the element McCool [16] constructs to show the $\ker \Phi_n$ is nontrivial for $n \geq 4$.

To understand this Vogt Relation better, recall the following for special linear characters of $F_n = \langle A, B, C, \ldots \rangle$, $n \geq 3$:

$$acb + abc - a \cdot bc - b \cdot ac - c \cdot ab + a \cdot b \cdot c = 0. \tag{6.2}$$

This is a way to write the non-Horowitz character acb as a polynomial in Horowitz generators. For $F_n = \langle A, B, C, D, \ldots \rangle$ $n > 4$, Vogt established (compare [5]):

$$\begin{aligned} 2abcd &= b \cdot d \cdot ac - a \cdot b \cdot cd - b \cdot c \cdot ad - ac \cdot bd \\ &\quad + ab \cdot cd + ad \cdot bc - d \cdot acb + a \cdot bcd + b \cdot acd + c \cdot abd. \end{aligned}$$

Notice, however, that again in this equation, acb is not a Horowitz generator. But by solving Equation 6.2 for acb and substituting into the last equation, we get a nontrivial quartic identity, which is the Vogt Relation (compare the following to Equation 6.1):

$$\begin{aligned} 2abcd &= a \cdot b \cdot c \cdot d - a \cdot b \cdot cd - a \cdot d \cdot bc - b \cdot c \cdot ad - c \cdot d \cdot ab \\ &\quad + a \cdot bcd + b \cdot acd + c \cdot abd + d \cdot abc \\ &\quad + ab \cdot cd - ac \cdot bd + bc \cdot ad \end{aligned} \tag{6.3}$$

There is an obvious generalization to this: To any basic word $WXYZ \in F_n$ where $w\ell(WXYZ) = r \geq 4$ and where $w\ell(W) = w\ell(X) = w\ell(Y) = 1$ (note the word-length of Z is $r-3$), one can use a direct substitution $W = A$, $X = B$, $Y = C$, and $Z = D$ into Equation 6.3 to get a relation among Horowitz generators in which all the terms on the right hand side are characters of basic words of length $r-1$ or less. One can recursively apply this construction until each Horowitz generator can be written as a polynomial (with rational coefficients) in the Sibirskii set. Adding these new relations to González and Montesinos [5] basis for $\mathcal{I}_n$ (one for each non-Sibirskii generator in the Horowitz set), we get a basis for the functions that vanish on $V_{F_n} \subset \mathbb{C}^{2^n-1}$.

Example 6.4. Go back to the case $n = 4$ in Example 6.3 and consider the González and Montesinos embedding of V_{F_4} into $\mathbb{C}^p$, where here $p = 14$. Essentially, this embedding neglects the Horowitz generator corresponding to the unique basic word of length 4 in F_4 as a coordinate. Here, $\mathcal{I}_4$ includes the polynomial in Equation 6.1. The Nielsen twist generator given in Equation 5.1 can be constructed as an endomorphism of $\mathbb{C}^{14}$:

$$\widehat{U}: \begin{array}{ccl} l & \mapsto & p \\ m & \mapsto & m \\ n & \mapsto & n \\ o & \mapsto & o \\ p & \mapsto & mp - l \\ q & \mapsto & v \\ r & \mapsto & w \\ s & \mapsto & s \\ t & \mapsto & t \\ u & \mapsto & u \\ v & \mapsto & mv - q \\ w & \mapsto & mw - r \\ x & \mapsto & z = \frac{1}{2}(lmno - lmu - los - mnr - nop + ly + mx + nw + ov + pu - qt + rs) \\ y & \mapsto & y \end{array}.$$

A quick calculation yields that

$$\det\left(Jac(\widehat{U})\right) = \frac{1}{2}m,$$

which vanishes along the $\{m = 0\}$-hypersurface. Hence the properties of volume preservation and non-zero Jacobian are not apparent for this embedding.

References

1. Culler, M., and Shalen, P., *Varieties of group representations and splittings of* 3*-manifolds,* Ann. Math (2) **117** (1983), 109-146.
2. Drensky, V., *Defining relations for the algebra of invarients of* 2×2 *matrices,* Alg. & Rep. Thy. **6** (2003), 193-214.
3. Florentino, C., *Invariants of 22 matrices, irreducible SL(2,C) characters and the Magnus trace map,* Geom.Dedicata **121** (2006), 167-186.
4. Fricke, R., and Klein, F., *Vorlesungen über die Theorie der automorphem Functionen,* Vol. 1, pp. 365-370. Leipzig: B.G. Teubner 1897. Reprint: New York Juhnson Reprint Corporation (Academic Press) 1965.
5. González-Acuña, F., and Montesinos-Amilibia, J.M., *On the character variety of group representations in* $SL(2, C)$ *and* $\mathbb{P}SL(2, C)$. , Math. Z. **214** (1993), no. 4, 627-652.
6. Goldman, W., *The symplectic nature of the fundamental groups of surfaces,* Adv. Math. **54** (1984), 200-225.
7. Goldman, W., *The complex-symplectic geometry of* $SL(2, \mathbb{C})$*-characters over surfaces,* Algebraic groups and arithmetic, 375-407, Tata Inst. Fund. Res., Mumbai, 2004.
8. Goldman, W., *Mapping class group dynamics on surface group representations,* Problems on mapping class groups and related topics, 189-214, Proc. Sympos. Pure Math. **74**, Amer. Math. Soc., Providence, RI, 2006.
9. Horowitz, R., *Characters of free groups represented in the two-dimensional special linear group,* Comm. Pure Appl. Math. **25** (1972), 635-649.
10. Horowitz, R., *Induced automorphisms on Fricke characters of free groups,* Trans. AMS **208** (1975), 41-50.
11. Huebschmann, J., *On the variation of the Poisson structures of certain moduli spaces,* Math. Ann. **319** (2001), 267-310.
12. Humphreys, J., *Reflection Groups and Coxeter Groups,* Cambridge: Cambridge University Press, 1990.
13. Kappraff, J., Jablan, S., Adamson, G., and Sazdanovich, R., *Golden Fields, Generalized Fibonacci Sequences, and Chaotic Matrices,* Forma **19** (2004), 367-387.

14. Koshy, T., *Fibonacci and Lucas numbers with applications,* Pure and Appl Math, New York: Wiley-Interscience, 2001.
15. Magnus, W., *Rings of Fricke characters and automorphism groups of free groups,* Math. Z. **170** (1980), 91-102.
16. McCool, J., *A faithful polynomial representation of* $Out(F_3)$*,* Math. Proc. Cambridge Philos. Soc. **106** (1989), no. 2, 207-213.
17. McCool, J., *Two-dimensional linear characters and automorphisms of free groups,* Combinatorial group theory (College Park, MD, 1988), 131-138, Contemp. Math. **109**, Amer. Math. Soc., Providence, RI, 1990.
18. Morgan, J., and Shalen, P., *Valuations, trees, and degenerations of hyperbolic structures I ,* Ann. Math. **120** (1984), 401-476.
19. Nielsen, J., *Die Isomorphismengruppe der freien Gruppen* Math. Ann. **91** (1924), 169-209.
20. Qi, D., *A note on irreducible, infinite Coxeter groups,* Expo. Math. **27** (2009), 87-91.
21. Sibirskii, K., *Algebraic invariants for a set of matrices,* Sib. Math. J., **9** (1968), no. 1, 115-124.
22. Thurston, W., *On the geometry and dynamics of diffeomorphisms of surfaces,* Bull. AMS **19** (1988), 417-431.
23. Vogt, H., *Sur les invariants fondamentaux des equations différentiellles linéaires du second order,* Ann. Sci. Ècol. Norm. Super., III. Ser. 6 (1889), 3-72.
24. Wang, S.S.-S., *A Jacobian criterion for separability,* J. Algebra **65** (1980), 453-440.
25. Weil, A., *Remarks on the cohomology of groups,* Ann. Math.(2) **80** (1964), 149-157.
26. Whittemore, A., *On special linear characters of free groups of rank* $n \leq 4$*,* Proc. AMS **40** (1973), 383-388.

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