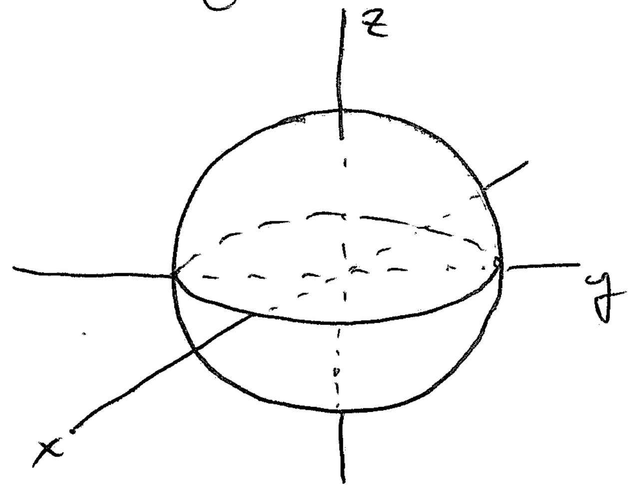
ex Find the volume of a sphere of radius ρ_0 in $\mathbb{R}^3$.



Strategy We first ~~show~~ calculate in rectilinear coordinates. Then we convert the coordinate system into ~~polar~~ spherical and recalculate.

Solution in rectilinear coordinates

We already saw that

$$\mathrm{Vol}(S^2(\rho_0)) = \int_{-\rho_0}^{\rho_0} \int_{-\sqrt{\rho_0^2 - x^2}}^{\sqrt{\rho_0^2 - x^2}} \int_{-\sqrt{\rho_0^2 - x^2 - y^2}}^{\sqrt{\rho_0^2 - x^2 - y^2}} 1 \, dz \, dy \, dx$$

Here, we do the individual steps in this calculation:

$$\mathrm{Vol}(S^2(\rho_0)) = \int_{-\rho_0}^{\rho_0} \int_{-\sqrt{\rho_0^2 - x^2}}^{\sqrt{\rho_0^2 - x^2}} \left(z \Big|_{-\sqrt{\rho_0^2 - x^2 - y^2}}^{\sqrt{\rho_0^2 - x^2 - y^2}} \right) dy \, dx$$

$$\mathrm{Vol}(S^2(\rho_0)) = \int_{-\rho_0}^{\rho_0} \underbrace{\int_{-\sqrt{\rho_0^2-x^2}}^{\sqrt{\rho_0^2-x^2}} 2\sqrt{\rho_0^2-x^2-y^2}\, dy}_{\text{inside integral}}\, dx$$

To make this next part easier, lets focus on the "inside" integral alone and let $\rho_0^2 - x^2 = a^2$ for $a \geq 0$ (why is this okay?). Then

$$\int_{-\sqrt{\rho_0^2-x^2}}^{\sqrt{\rho_0^2-x^2}} 2\sqrt{\rho_0^2-x^2-y^2}\, dy = \int_{-a}^{a} 2\sqrt{a^2-y^2}\, dy.$$

Here, let $y = a\sin t$, with $dy = a\cos t\, dt$ be our trig substitution. Then

$$2\int_{-a}^{a} \sqrt{a^2-y^2}\, dy = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2-a^2\sin^2 t}\; a\cos t\, dt$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos^2 t\, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2(1+\cos 2t)\, dt$$

$$= a^2\left(t + \tfrac{1}{2}\sin 2t\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi a^2 = \pi(\rho_0^2 - x^2)$$

Hence

$$\mathrm{Vol}(S^2(\rho_0)) = \int_{-\rho_0}^{\rho_0}\int_{-\sqrt{\rho_0^2-x^2}}^{\sqrt{\rho_0^2-x^2}} 2\sqrt{\rho_0^2-x^2-y^2}\,dy\,dx = \int_{-\rho_0}^{\rho_0} \pi(\rho_0^2-x^2)\,dx$$

And we finish

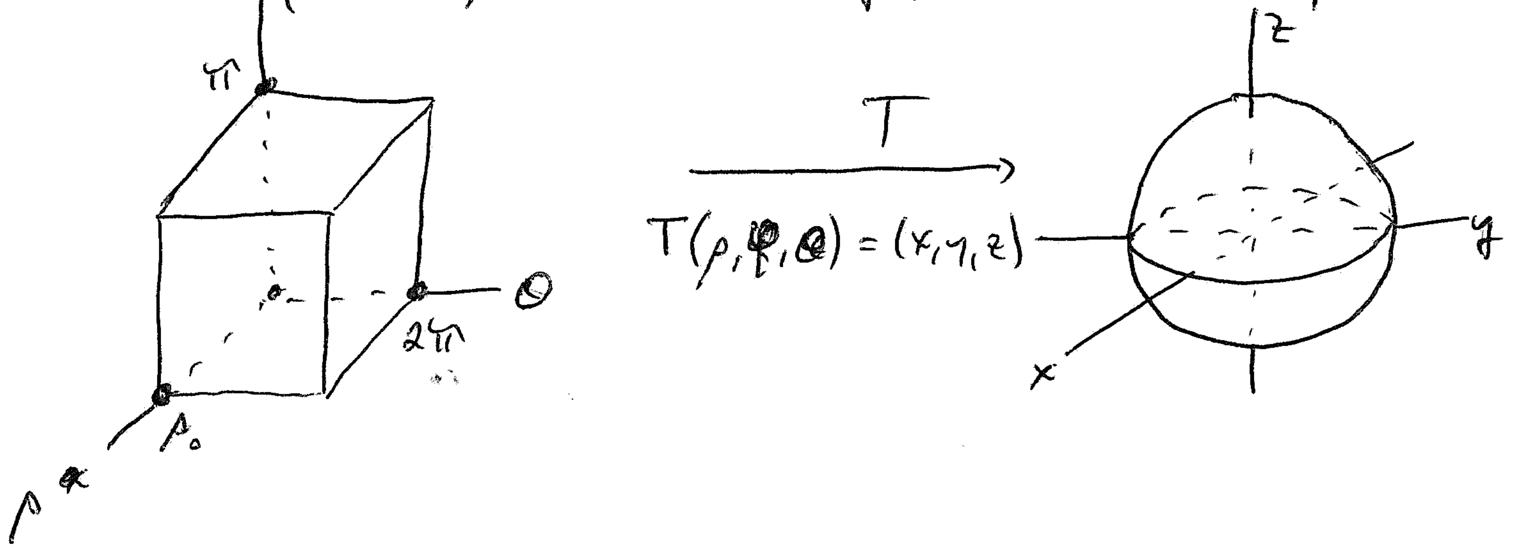
$$\mathrm{Vol}(S^2(\rho_0)) = \int_{-\rho_0}^{\rho_0} (\pi\rho_0^2 - \pi x^2)\,dx = \left(\pi\rho_0^2 x - \pi\frac{x^3}{3}\Big|_{-\rho_0}^{\rho_0}\right)$$

$$= \pi\rho_0^3 - \pi\frac{\rho_0^3}{3} + \pi\rho_0^3 - \pi\frac{\rho_0^3}{3} = \boxed{\frac{4}{3}\pi\rho_0^3}$$

Solution in spherical coordinates

First, lets do the transformation: Let

$$x = \rho\sin\varphi\cos\theta,\quad y = \rho\sin\varphi\sin\theta,\quad z = \rho\cos\varphi$$



The Jacobian of this transformation is

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$= \cos\varphi \begin{vmatrix} \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix}$$

$$-\left(-\rho\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix}\right)$$

$$= \rho^2\cos\varphi\left(\sin\varphi\cos\varphi\cos^2\theta + \sin\varphi\cos\varphi\sin^2\theta\right)$$

$$+ \rho^2\sin\varphi\left(\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta\right)$$

$$= \rho^2\sin\varphi\cos^2\varphi + \rho^2\sin\varphi\sin^2\varphi$$

$$= \rho^2\sin\varphi$$

Hence

$$Vol(S^2(\rho)) = \int_{-\rho_0}^{\rho_0}\int_{-\sqrt{\rho_0^2-x^2}}^{\sqrt{\rho_0^2-x^2}}\int_{-\sqrt{\rho_0^2-x^2-y^2}}^{\sqrt{\rho_0^2-x^2-y^2}} dz\,dy\,dx = \int_0^{\rho_0}\int_0^{\pi}\int_0^{2\pi} \left|\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}\right| d\theta\, d\varphi\, d\rho$$

$$= \int_0^{\rho_0}\int_0^{\pi}\int_0^{2\pi} \rho^2 \sin\varphi \, d\theta\, d\varphi\, d\rho = \int_0^{\rho_0}\int_0^{\pi}\left(\rho^2 \sin\varphi)\theta\Big|_0^{2\pi}\right) d\varphi\, d\rho$$

$$= 2\pi \int_0^{\rho_0}\int_0^{\pi} \rho^2 \sin\varphi \, d\varphi\, d\rho = 2\pi \int_0^{\rho_0} \rho^2 \left(-\cos\varphi\Big|_0^{\pi}\right) d\rho$$

$$= 2\pi \int_0^{\rho_0} 2\rho^2 \, d\rho = 4\pi \frac{\rho^3}{3}\Big|_0^{\rho_0} = \frac{4}{3}\pi \rho_0^3$$