Sept 7, 2016: Lecture 1. Section 2.1

A function $f: X \to Y$ from a set X to a set Y is the same as what you already know:

- Assignment of a unique $y \in Y$ to each $x \in X$.
- X is the domain, Y the codomain
- $f(X) \subset Y$ (as a set) is the range, called the image of X under f.
- For $Z \subset Y$, $f^{-1}(Z) \subset X$ is called the inverse image (as a set) of Z, or the preimage. $f^{-1}(y) = \phi$, when $y \notin f(X)$.

bijective if both

- Is 1-1 (one-to-one, or injective) if $\#\{x \in X \mid f(x) = y\} \leq 1 \quad \forall y \in Y$
- Is onto (or surjective) if $\forall y \in Y$, $y = f(x)$ for at least one $x \in X$.

For this class X, Y will be subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$.

Special Nomenclature and notation

- If $Y \subset \mathbb{R}$, f is called real-valued, or scalar-valued.
- If $Y \subset \mathbb{R}^n$, $n \geq 1$, called vector-valued.

Note: Vector-valued functions have expressions that are real-valued on each coordinate.

- Where important, $x \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^n$, f is real-valued, $\vec{f}$ is vector-valued.

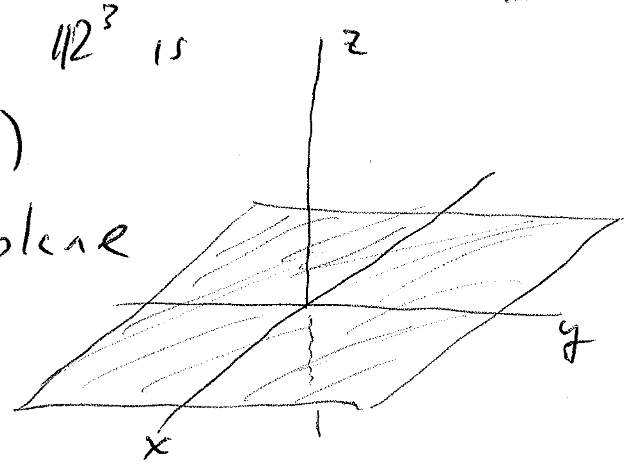
Note: If not important, will leave off the hat.

- Def. A map $p: X \to X$ is called a projection if $p(p(x)) = p(x)$ $\forall x \in X$.
  - $p(X) \subset X$ is called the projection of X onto $p(X)$, a subspace.
  - $p|_{p(X)} = Id_{p(X)}$.
  - In a product space, $p_i: X \times \cdots \times X \circlearrowleft$ where $p_i(\vec{x}) = p_i((x_1, \dots, x_n)) = (0, \dots, 0, x_i, 0, \dots, 0)$ is the ith projection.
  - Sometimes write $p_i(\vec{x}) = x_i$ but this is not quite correct.

ex. A common projection in $\mathbb{R}^3$ is

$$(x,y,z) \longmapsto (x,y,0)$$

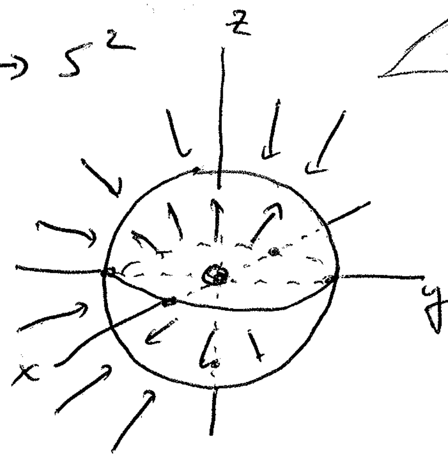
projection onto the xy-plane



ex. $r: \mathbb{R}^3 - \vec{0} \longrightarrow S^2$

$$r(\vec{x}) = \frac{\vec{x}}{\|\vec{x}\|}$$

This is the vector normalization function.



$S^2 = \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x}\| = 1\}$.

Do you see why $\vec{0}$ cannot be in the domain?

Visualizing and studying functions of and/or to $\mathbb{R}^n$, $n>1$, is tricky.

Some tools are:

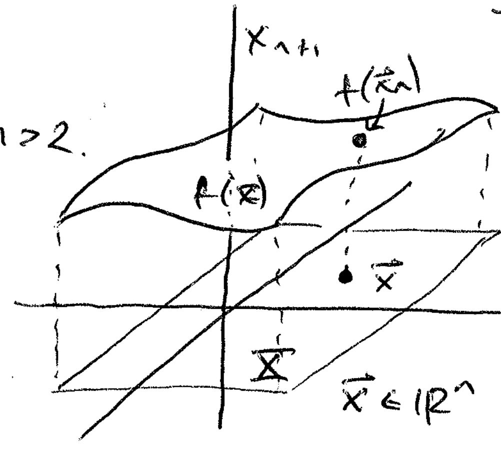
(I) <u>Graphs</u> – a visual depiction of a functional relationship via point by point plotting of solution sets.

This is best for real-valued functions as a height over floor schematic.

(I) Cont'd. For $f : X \subset \mathbb{R}^n \to \mathbb{R}$,

$$\text{graph}(f) = \{ (\vec{x}, f(\vec{x})) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \mid x_{n+1} = f(\vec{x}) \}$$

- Usable for $X \subset \mathbb{R}^2$
- Not so usable for $X \subset \mathbb{R}^n$, $n > 2$.
- Generalizes visualization in Calculus I - II.
- Generally speaking, the dimension of $f(X)$ will be the same as that of X.



- $\text{graph}(f) \subset \mathbb{R}^{n+1}$ always projects to X by $(x_1, \dots, x_n, f(\vec{x})) \longmapsto (x_1, \dots, x_n, 0)$.

For $\vec{g} : X \subset \mathbb{R}^2 \to \mathbb{R}^2$, $g(\vec{x}) = g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} g_1(\vec{x}) \\ g_2(\vec{x}) \end{bmatrix}$

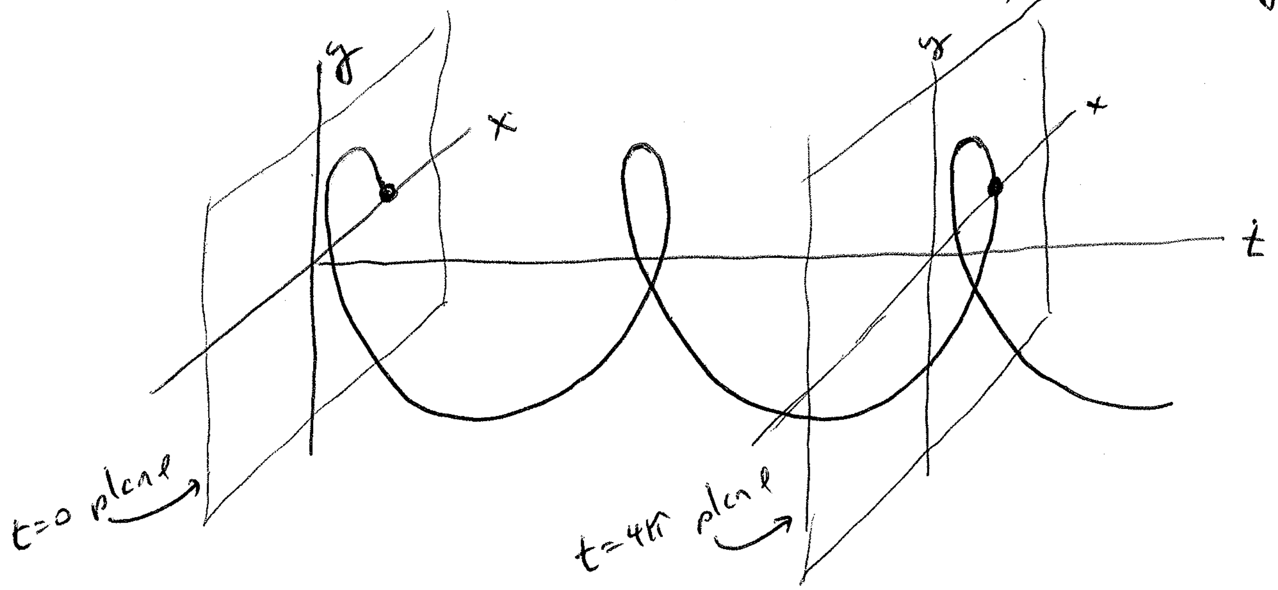
where each $g_i : X \to \mathbb{R}$ is real valued.

Here $\text{graph}(\vec{g}) = \left\{ (x, y, z, u) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} z = g_1(x, y) \\ u = g_2(x, y) \end{array} \right\}$ and is hard to visualize.

(I) cont'd.

But $\vec{h}: \mathbb{R} \to \mathbb{R}^2$, $h(t) = (\cos t, \sin t)$ can be graphed:

$$\text{graph}(\vec{h}) = \{(t, x, y) \in \mathbb{R}^3 \mid x = x(t),\ y = y(t)\}$$



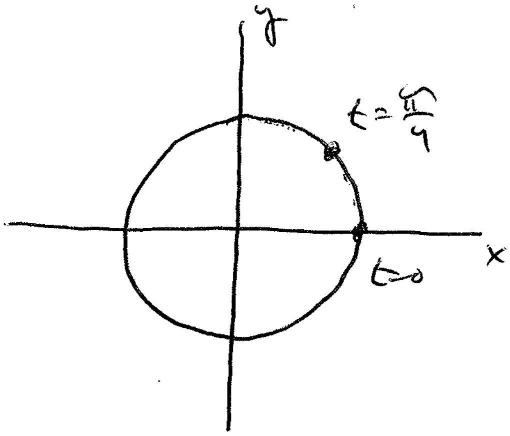
Doable, but still hard to see and analyze.

(II) Parameterizations: Generalized coordinates placed on a subset of $\mathbb{R}^n$ through continuous fncs so that pts are distinguishable via parameter values instead of ambient coordinates.

- Allows one to describe a subset of $\mathbb{R}^n$ by a smaller # of variables

The dimension of a parameterization is the # of coordinates needed to distinguish pts.

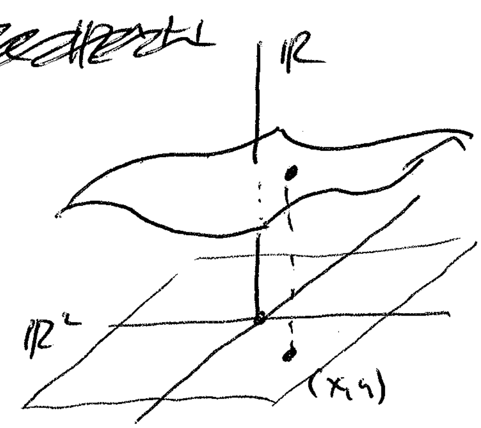
ex. $\vec{h}:\mathbb{R}\to\mathbb{R}^2$, $\vec{h}(t)=(\cos t,\sin t)\in\mathbb{R}^2$ parameterizes the unit circle in $\mathbb{R}^2$.



Here, t is a coordinate directly on circle and param. is 1-d.

Note: Parameterizations are 1-1 but only locally is necessary.

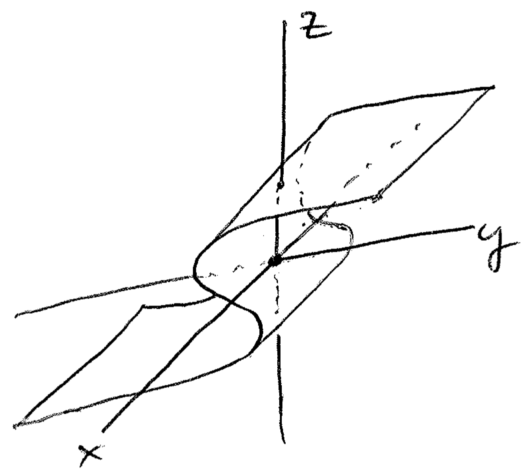
ex. The domain of a graph of $f: X\subset\mathbb{R}^n\to\mathbb{R}$ always parameterizes $\text{graph}(f)\subset\mathbb{R}^{n+1}$. why?



But it can parameterize subsets of $\mathbb{R}^n$ that are not graphs of functions

ex. Let $D=[-2,2]\times[-1.5,1.5]$ and $\Phi: D\to\mathbb{R}^3$,

$$\Phi(u,v)=\left(u,\ \frac{3(v^3-v)}{4},\ \frac{2v}{5}\right)$$



Slices, Sections of graphs of functions.

Def Let $f: X \subset \mathbb{R}^n \to \mathbb{R}$ be real-valued.

(a) A c-level set of f is

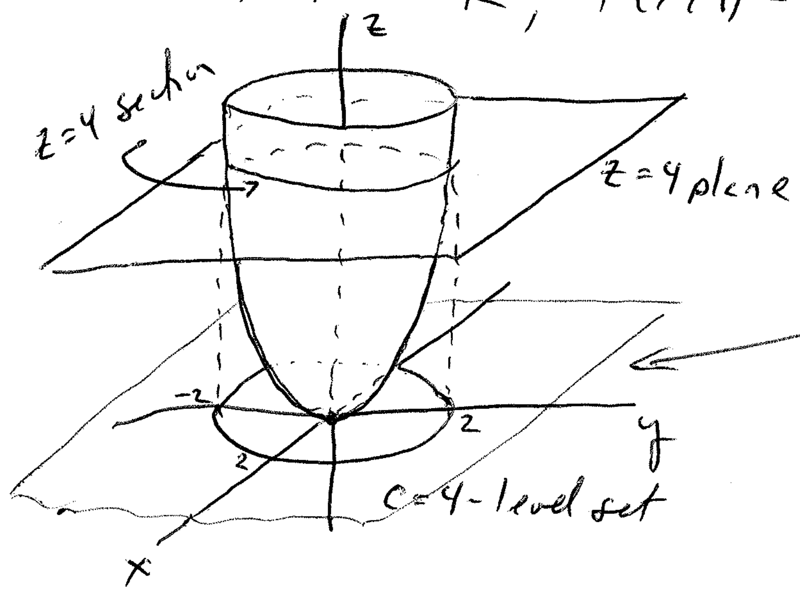
$$\{\vec{x} \in X \mid f(\vec{x}) = c\}$$

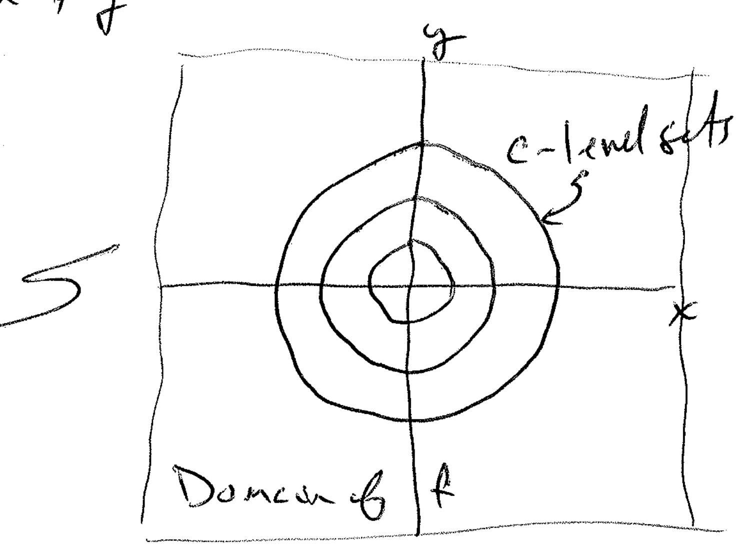
(b) A horizontal section of f at c is

$$\{(\vec{x}, c) \in \text{graph}(f) \subset \mathbb{R}^{n+1} \mid c = f(\vec{x})\}.$$

- it is the graph of a c-level set
- Also called a c-contour set (but not a c-section!)

ex. $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + y^2$





topographic map

Notes ① All level sets here look like $c = x^2+y^2$ (circles in $\mathbb{R}^2$ of radius $\sqrt{c}$).

② c-level sets are projections of hor. sections back down into the domain.

③ Can also write a c-level set as the inverse image (as a set) of a value $c \in \mathbb{R}$.

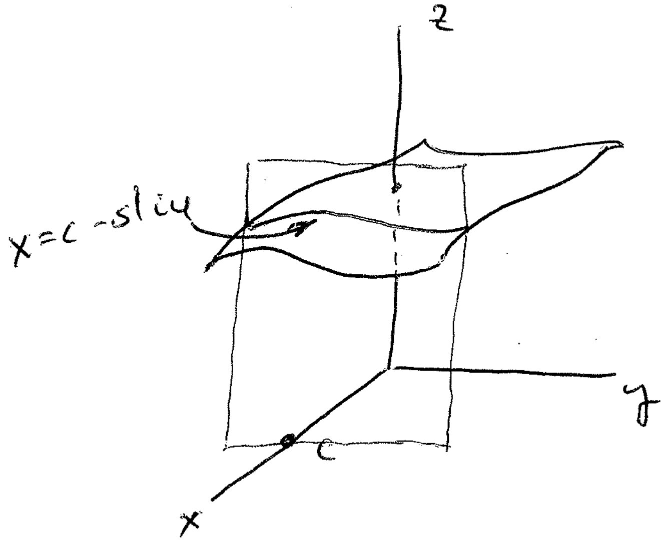
$$c\text{-level set} = f^{-1}(c) \subset X.$$

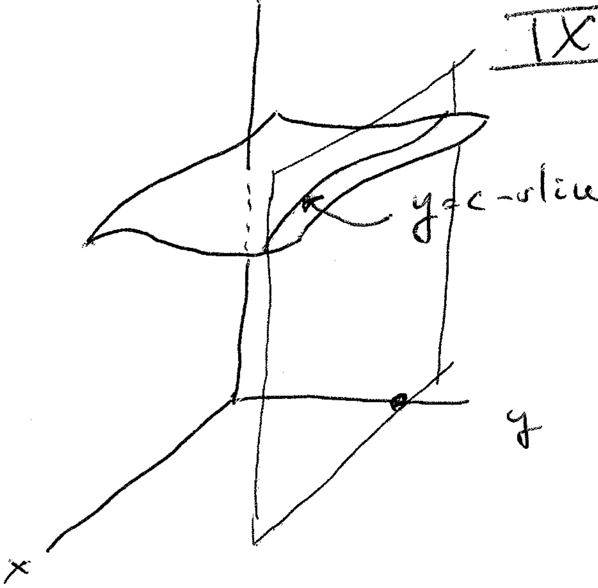
Ⓒ A vertical section (or slice) of graph(f) is the intersection of graph(f) with a vertical subspace of $\mathbb{R}^{n+1}$ formed by setting one of the domain coordinates to a constant.

For $\text{graph}(f) = \{(x_1, \cdots, x_n; z) \in \mathbb{R}^{n+1} \mid f(\vec{x}) = z\}$

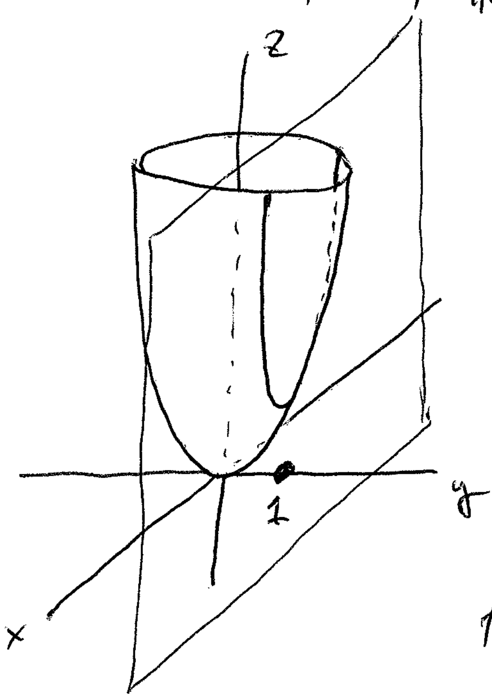
the x_i-slice at c is the set

$$\{(\vec{x}, z) \in \mathbb{R}^{n+1} \mid z = f(x_1, \cdots, x_{i-1}, c, x_{i+1}, \cdots, x_n)\}$$





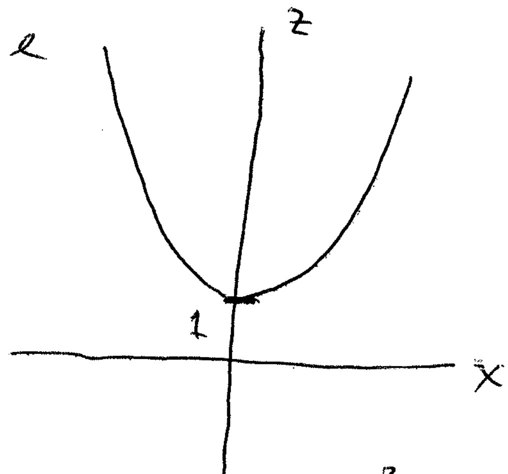
ex. Back to $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = z = x^2+y^2$



Here, the $y=1$-slice is

graph$(f) \cap$ (xz-plane at $y=1$)

The actual slice is given by the equation $z = x^2+1$ inside the xz-plane



Here the $y=1$-slice is parameterized by x and is the set

$\{(x,1,x^2+1) \in \mathbb{R}^3\}$ as a curve in $\mathbb{R}^3$.