What is a limit?

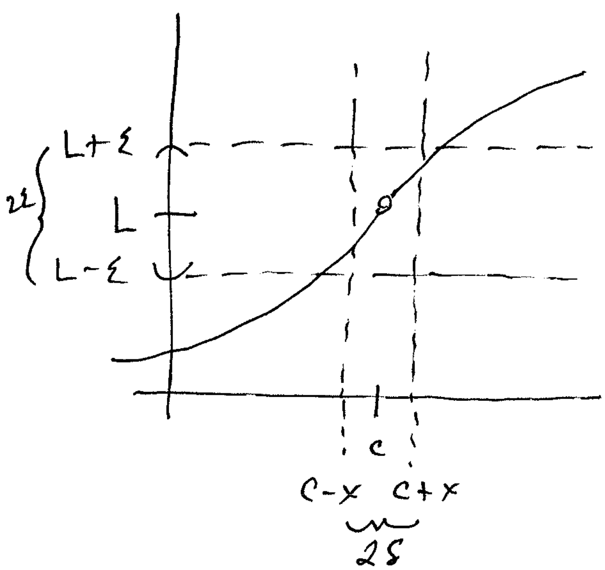
In Calculus I, we have:

<u>Def</u> For $f: I \subset \mathbb{R} \to \mathbb{R}$, f has a limit L at $x = c$, denoted

$$\lim_{x \to c} f(x) = L$$

if $\forall \varepsilon > 0$, $\exists \delta > 0 \ni$ if $0 < |x - c| < \delta$,

then $|f(x) - L| < \varepsilon$.



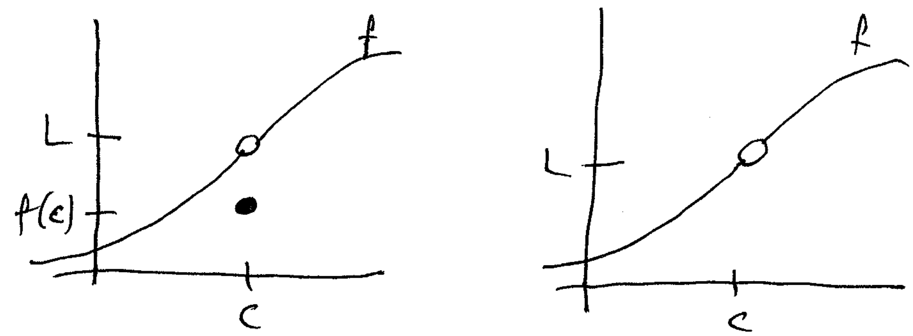
<u>Notes</u> ① Defining a limit <u>at</u> c gives a notion of what happens near c!

② Limits have no concern of what happens <u>at</u> c

③ If, any time I define a small $\underbrace{\text{interval}}_{\varepsilon}$ around a number L, I can find a small $\underbrace{\text{interval}}_{\delta}$ of inputs whose fnc values stay in the function interval, then near $x = c$, function values are near L.

(4) Limits exist even when function values at c are different or nonexistent:

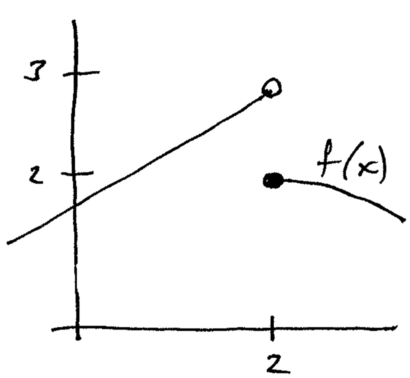
$\lim_{x \to c} f(x) = L$ for both.



(5) The idea of x approaching c involves only 2 possible directions.

$\lim_{x \to c^+} f(x)$ $\qquad$ $\lim_{x \to c^-} f(x)$

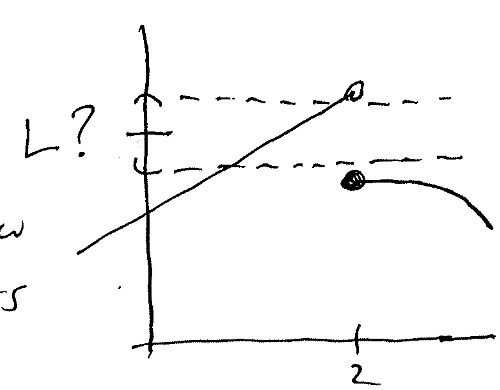
Only when each exist and are equal can the limit actually exist.



Here, $\lim_{x \to 2^-} f(x) = 3 \neq 2 = \lim_{x \to c^+} f(x)$

Hence $\lim_{x \to 2} f(x)$ DNE.

But still try to find an L and a small ε-window where all fnc values of pts near $x=2$ reside....



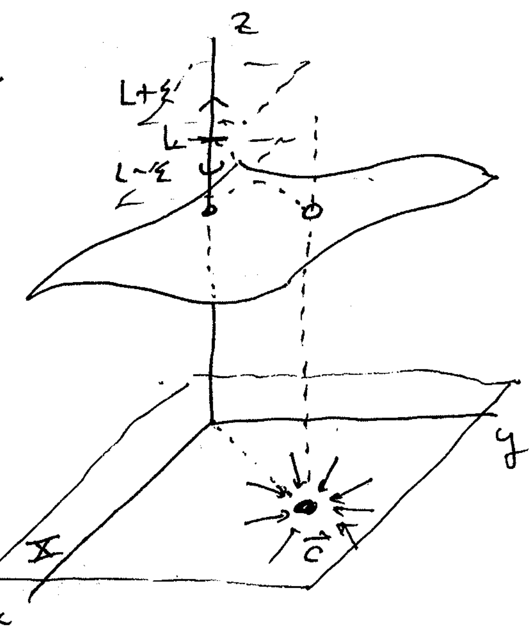
In this course, we have $\vec{f}: X \subset \mathbb{R}^n \to \mathbb{R}^m$, with $\vec{x} \in \mathbb{R}^n$, $\overrightarrow{f(\vec{x})} \in \mathbb{R}^m$.

But def is still the same:

<u>Def</u> $\vec{f}$ has a limit $\vec{L}$ at $\vec{x} = \vec{c}$, denoted

$$\lim_{\vec{x} \to \vec{c}} \overrightarrow{f(\vec{x})} = \vec{L}$$

if $\forall \varepsilon > 0, \exists \delta > 0 \ni$ if $0 < \|\vec{x} - \vec{c}\| < \delta$, then $\|\overrightarrow{f(\vec{x})} - \vec{L}\| < \varepsilon$.



Here $\|\vec{x}\|$ is Euclidean Norm, and $\|\vec{x} - \vec{y}\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

Notice all of the similarities but one big difference:

The number of ways to approach $\vec{c}$ in the domain makes things much more complicated!!

First, some topology: The Euclidean metric on $\mathbb{R}^n$ allows for a nice definition of "open" set:

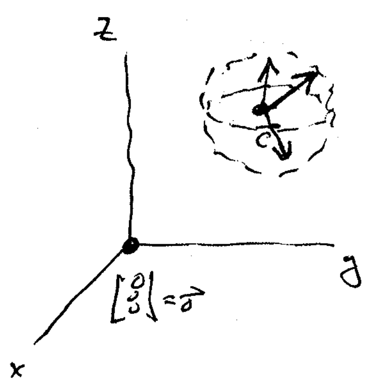
Topology of $\mathbb{R}^n$:

Def An open ball of radius $\varepsilon > 0$ centered at $\vec{c} \in \mathbb{R}^n$ is $B_\varepsilon(\vec{c}) = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{c}\| < \varepsilon\}$.

Notes: ① In $\mathbb{R}^3$, this is the usual ball you played with as a kid, but without the skin!

② In $\mathbb{R}^2$, it is the disc of radius ε w/o the circle edge. In $\mathbb{R}^1$? In $\mathbb{R}^4$?

③ One can think of it as the set of all vectors of length less than ε based at $\vec{c}$ (and not $\vec{0}$!)



Def. A closed ε-ball is

$$\overline{B_\varepsilon}(\vec{c}) = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{c}\| \leq \varepsilon\}$$

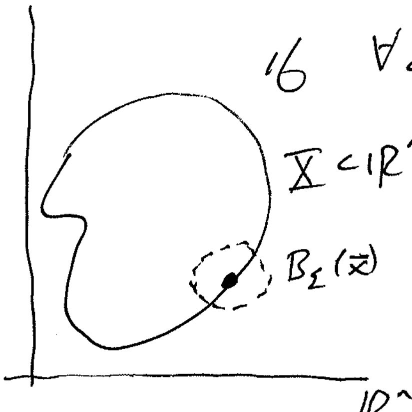
And the skin, by itself ($\{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{c}\| = \varepsilon\}$)? What does this look like?

The skin is the boundary of both $B_\varepsilon(\vec{c})$ and $\overline{B_\varepsilon}(\vec{c})$.

Def A set $X \subset \mathbb{R}^n$ is called "open" if

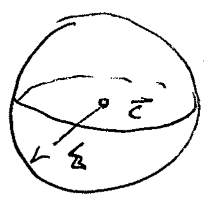
$$\forall x \in X,\ \exists\, \varepsilon > 0 \ni B_\varepsilon(\vec{x}) \subset X.$$

Def A pt $x \in \mathbb{R}^n$ is a boundary pt of $X \subset \mathbb{R}^n$ if $\forall \varepsilon > 0$, $B_\varepsilon(\vec{x})$ contains pts in X and pts not in X.

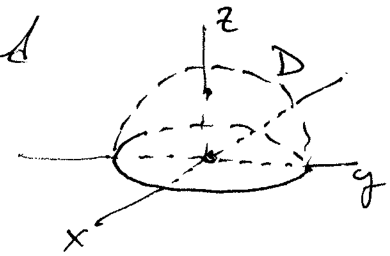


Def A set $X \subset \mathbb{R}^n$ is called closed if it contains all of its boundary pts

$B_\varepsilon(\vec{c})$ is open



$\overline{B_\varepsilon(\vec{c})}$ is closed

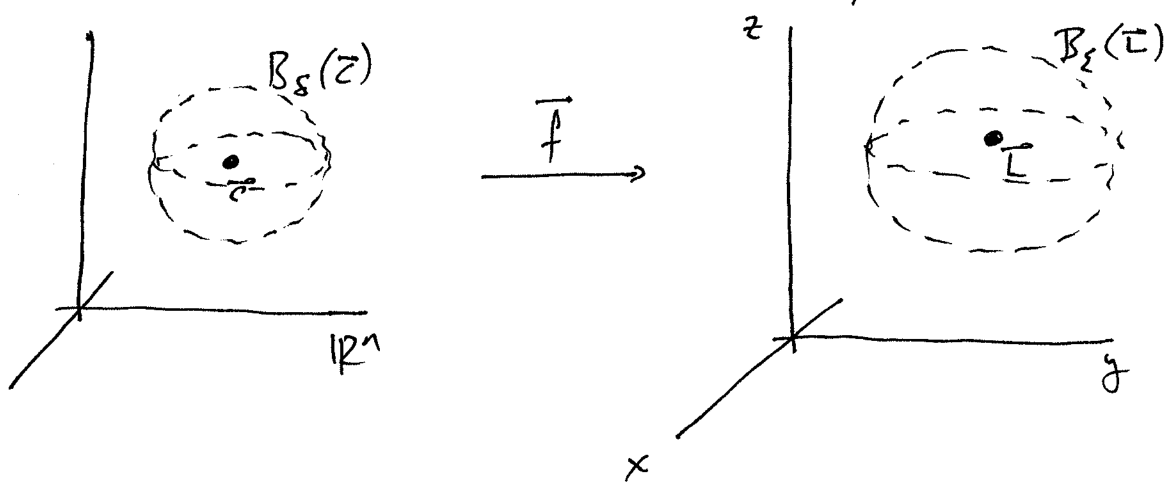


$D = \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x}\| < \varepsilon \text{ and } z \geq 0\}$. is neither open nor closed. (contains some of its boundary pts).

Def. Given $X \subset \mathbb{R}^n$, a pt $x \in X$ where $\exists\, \varepsilon > 0 \ni B_\varepsilon(\vec{x}) \subset X$ is called an interior pt.

Note: Given $X \subset \mathbb{R}^n$ and an interior pt $x \in X$, any open subset $U \subset X$ where $x \in U$ is called a neighborhood of x in X, denoted $U(x) \subset X$.

A better way to "see" a limit w/o a graph:
Separate the domain and codomain spaces.



$\vec{f}$ has a limit $\vec{L}$ at $\vec{x}=\vec{c}$ if given any $B_\varepsilon(\vec{L})$, you can find a δ-ball $B_\delta(\vec{c})$ so that image of δ-ball is entirely inside the ε-ball; or $\exists\,\delta>0$ so that $\vec{f}(B_\delta(\vec{c})) \subset B_\varepsilon(\vec{L})$.

In practice, ① limits are hard to calculate using definition for pathological functions.
② Follow all the Calculus I limit laws (pg 106).

③ Most functions in vector calculus (like in Calc I) are "nice": They behave well on their full domain
- Vector-valued functions are scalar valued on each component
- Scalar-valued fncs, like in Calc I are still "nice"

ex. $f(x,y) = \cos(x+y)$ will still have limits everywhere due to the same reasons it did in calculus I.

Same with polynomials, exponentials, rational fncs, logarithms, etc.

And, for point ① above, keep in mind:

ⓒ One technique to study limits is to reduce the approach of $\vec{x}$ to $\vec{c}$ to one direction, and use techniques from Calc I:

ex. $f(x,y) = \frac{xy}{x^2+y^2}$. Not defined at $(0,0)$. Does limit exist?

Explore: • Along x-axis, $y=0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) \underset{y=0}{=\!=} \lim_{(x,0)\to(0,0)} \frac{x \cdot 0}{x^2} = 0$$

• Same along y-axis!

• Along line $y=x$?

$$\lim_{(x,y)\to(0,0)} f(x,y) \underset{y=x}{=\!=} \lim_{(x,x)\to(0,0)} \frac{x \cdot x}{x^2+x^2} = \lim_{x\to 0} \left(\frac{1}{1+1}\right) = \frac{1}{2}.$$

If approaching from different directions yields different results, can a limit exist??

② Another technique: Switch to polar coordinates and use the fact that $B_\varepsilon(\vec{x}) = B_\rho(\vec{x})$ where ρ is the "distance" coordinate.

ex: $f(x,y) = \frac{xy}{x^2+y^2}$. Here $x = \rho\cos\theta$, $y = \rho\sin\theta$.

$$f(\rho,\theta) = \frac{(\rho\cos\theta)(\rho\sin\theta)}{\rho^2\cos^2\theta + \rho^2\sin^2\theta} = \cos\theta\sin\theta$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(\rho,\theta)\to(0,\theta)} \cos\theta\sin\theta$$

Here limit will again be different for different approach directions (lines of constant θ).

See Mathematica

Continuity - Same concept as Calc I with some additional structure:

Def For $\vec{f}: X \subset \mathbb{R}^n \to \mathbb{R}^m$, $\vec{f}$ is said to be continuous at $\vec{a}$ if either $\vec{a}$ is an isolated pt of X, or if $\lim_{\vec{x}\to\vec{a}} \overline{f(\vec{x})} = \overline{f(\vec{a})}$

$\vec{f}$ is a continuous fnc on X if it is continuous at all pts of X.

Notes (1) Graphs won't have tears or cliffs or breaks in it. (sort of).

(2) Like in Calc I, sums and scalar multiples of continuous fncs are continuous.

(3) Also for products and quotient when they make sense.

(4) Composites? if $\vec{f}: X \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{g}: Y \subset \mathbb{R}^m \to \mathbb{R}^p$ are continuous, and $f(X) \subset Y$, then $\overrightarrow{g \circ f}: X \subset \mathbb{R}^n \to \mathbb{R}^p$ is cont.

(5) $\vec{f}: X \subset \mathbb{R}^n \to \mathbb{R}^m$ is cont. at $\vec{a}$ iff each $f_i: X \subset \mathbb{R}^n \to \mathbb{R}$ is cont. at $\vec{a}$. $i = 1, \dots, m$