Section 2.4

Rules for differentiation?

- They always look like Calculus I rules..., when they make sense!
- Since the derivative of a func in Calc III is a matrix (of functions) (of numbers at a point), making new functions via algebra or composition must be compatible.

(I) Constant Multiple Rule is always the same:

For $\vec{f}: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $c \in \mathbb{R}$, if $\vec{f}$ is diff, then $(c\vec{f}): X \subset \mathbb{R}^n \to \mathbb{R}^m$, is diff and $D(c\vec{f})(\vec{x}) = c\, D\vec{f}(\vec{x})$.

(II) Sum and Diff rule only requires domain & codomain to be the same:

For $\vec{f}, \vec{g}: X \subset \mathbb{R}^n \to \mathbb{R}^m$, $\vec{h} = \vec{f} + \vec{g}$.

$$D\vec{h}(\vec{x}) = D(\vec{f}+\vec{g})(\vec{x}) = D\vec{f}(\vec{x}) + D\vec{g}(\vec{x})$$

(III) Products are trickier, since codomains ~~may~~ be multidimensional:

Let $\vec{f}: X \subset \mathbb{R}^n \to \mathbb{R}^m$, $\vec{g}: X \subset \mathbb{R}^n \to \mathbb{R}^p$, and

Let $\vec{h} = \vec{f} \cdot \vec{g}$.

III cont'd.

Here, the "dot" in $\vec{h} = \vec{f}(\vec{x}) \cdot \vec{g}(x)$ needs to make sense. When it does, then the Product Rule from Calc I still holds:

$$D\vec{h}(\vec{x}) = D(\vec{f} \cdot \vec{g})(\vec{x}) = D\vec{f}(\vec{x}) \cdot g(\vec{x}) + \vec{f}(\vec{x}) \cdot D\vec{g}(\vec{x})$$

ex. $p = 1$: Here $g(\vec{x})$ is a scalar, and $Dg(\vec{x})$ is a $1 \times n$ matrix, so

$$\underbrace{D\vec{h}(\vec{x})}_{m \times n} = \underbrace{D\vec{f}(\vec{x})}_{m \times n} \cdot \underbrace{g(\vec{x})}_{\text{scalar}} + \underbrace{\vec{f}(\vec{x})}_{m \times 1} \underbrace{D\vec{g}(\vec{x})}_{1 \times n} \quad \checkmark$$

example: $\vec{f}(x,y,z) = \begin{bmatrix} xy + y^2 z \\ x^4 z \end{bmatrix}$, $g(x,y,z) = \ln(yz)$

$$\vec{h}(x,y,z) = \vec{f}(x,y,z) \cdot g(x,y,z) = \begin{bmatrix} (xy + y^2 z)\ln(yz) \\ (x^4 z)\ln(yz) \end{bmatrix}$$

$$D\vec{f}(\vec{x}) = \begin{bmatrix} y & x + 2yz & y^2 \\ 4x^3 z & 0 & x^4 \end{bmatrix} \qquad Dg(\vec{x}) = \begin{bmatrix} 0 & \frac{1}{y} & \frac{1}{z} \end{bmatrix}$$

Now put it all together and show equality.

$$D\vec{f}(\vec{x}) \cdot g(\vec{x}) + \vec{f}(\vec{x}) \cdot Dg(\vec{x}) =$$

$$\begin{bmatrix} y & x + 2yz & y^2 \\ 4x^3 z & 0 & x^4 \end{bmatrix} \ln(yz) + \begin{bmatrix} xy + y^2 z \\ x^4 z \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{y} & \frac{1}{z} \end{bmatrix}.$$

$$= \begin{bmatrix} y\ln(yz) & (x + 2yz)\ln(yz) & y^2 \ln(yz) \\ 4x^3 z \ln(yz) & 0 & x^4 \ln(yz) \end{bmatrix} + \Bigg[\cdots$$

ex. dot product ($m = p \geq 1$): if $h(\bar{x}) = \vec{f}(\bar{x}) \circ \vec{g}(\bar{x})$

dot product.

Then $h : \bar{X} \subset \mathbb{R}^n \to \mathbb{R}$, and

$$Dh(\bar{x}) = [D_{x_1} h(\bar{x}) \quad \cdots \quad D_{x_n} h(\bar{x})], \text{ with}$$

$$D_{x_i} h = \frac{\partial}{\partial x_i} h(\bar{x}) = \frac{\partial}{\partial x_i} \sum_{j=1}^{m} f_j(\bar{x}) \circ g_j(\bar{x})$$

Sum Rule
$$= \sum_{j=1}^{m} \frac{\partial}{\partial x_i} \left(f_j(\bar{x}) g_j(\bar{x}) \right) \quad \text{Sum Rule}$$

Calc I Product Rule
$$= \sum_{j=1}^{m} \frac{\partial f_j}{\partial x_i}(\bar{x})\, g_j(\bar{x}) + f_j(\bar{x}) \frac{\partial g_j}{\partial x_i}(\bar{x})$$

Sum Rule
$$= \underbrace{\sum_{j=1}^{m} \underbrace{\frac{\partial f_j}{\partial x_i}(\bar{x})}_{\substack{i\text{th col} \\ \text{of } D\vec{f}(\bar{x})}}}_{1 \times m} \underbrace{g_j(\bar{x})}_{1 \times m} + \underbrace{\sum_{j=1}^{m} f_j(\bar{x})}_{1 \times m} \underbrace{\underbrace{\frac{\partial g_j}{\partial x_i}(\bar{x})}_{\substack{i\text{th col of} \\ D\vec{g}(\bar{x})}}}_{1 \times m}$$

$$= D_{x_i} \vec{f}(\bar{x}) \cdot \vec{g}(\bar{x}) + \vec{f}(\bar{x}) \cdot D_{x_i} \vec{g}(\bar{x})$$

dot products

Special Note: Careful with products! The 2 above are symmetric: $f(\bar{x}) \cdot g(\bar{x}) = g(\bar{x}) \cdot f(\bar{x})$.

If the product is not symmetric, then neither is the Product Rule:

ex. Cross product in $\mathbb{R}^3$: $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ in general.

actually antisymmetric $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Hence for $\vec{f}, \vec{g} : X \subset \mathbb{R}^n \to \mathbb{R}^3$, and $\vec{h}(\vec{x}) = \vec{f}(\vec{x}) \times \vec{g}(x)$

$$D\vec{h}(\vec{x}) = D(\vec{f} \times \vec{g})(\vec{x})$$
$$= D\vec{f}(\vec{x}) \times \vec{g}(x) + \vec{f}(x) \times D\vec{g}(x)$$

where $Df : X \subset \mathbb{R} \to \mathbb{R}^3$
Dg.

Similar notions for Quotient Rule.

<u>A note on partial derivatives</u>

Given a real-valued $f : X \subset \mathbb{R}^n \to \mathbb{R}$, then

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} : X \longrightarrow \mathbb{R}.$$

If all <u>exist</u> and are <u>continuous</u> in a nbhd of $\vec{x}$, then we say f is of class C^1, or $f \in C^1$.

Def A second partial of f wrt a variable x, say, is one of

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) : X \to \mathbb{R}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) : X \to \mathbb{R}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) : X \to \mathbb{R}$$

Also written $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial z \partial x}$, or f_{xx}, f_{xy}, f_{xz}

notice the order

If all of exist and are continuous in a nbhd of every pt $\bar{x} \in \mathbb{X}$, we say $f \in C^2$.

(#)

Let $f: \mathbb{X} \subset \mathbb{R}^n \to \mathbb{R}$. For $i_1, \dots, i_k \in \{1, \dots, n\}$ the kth partial derivative of f with respect to $x_{i_1}, \dots, x_{i_k}$ is

$$\frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_1}}(\bar{x}) = \frac{\partial}{\partial x_{i_k}}\left(\cdots \left(\frac{\partial f}{\partial x_{i_1}}(\bar{x}) \right) \cdots \right)$$

or we write $f_{x_{i_1} \cdots x_{i_k}}$.

ex $f(x,y,z) = z\cos(2xy)$

$f_x(x,y,z) = -z\sin(2xy)2y = -2yz\sin(2xy)$

$f_y(x,y,z) = -2xz\sin(2xy)$

$f_z(x,y,z) = \cos(2xy)$

$f_{xy}(x,y,z) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) =$

$= -2z\sin(2xy) - 4xyz\cos(2xy)$

$f_{yx}(x,y,z) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$= -2z(2xy) - 4xyz\cos(2xy)$

Notice these are equal!

It turns out, this is always true when $f \in C^2$:

<u>Thm</u> Suppose $f: X \subset \mathbb{R}^n \to \mathbb{R}$ is C^2 on an open X. Then for any choice of $i_1, i_2 \in \{1, \cdots, n\}$,

$$\frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}} = \frac{\partial^2 f}{\partial x_{i_2} \partial x_{i_1}}$$

The proof is constructive. We won't do it in class.

<u>Def</u> $f: X \subset \mathbb{R}^n \to \mathbb{R}$ is of class C^n, $n \in \mathbb{N}$ if it has continuous partial derivatives up to and including order n. $\vec{g}: X \subset \mathbb{R}^n \to \mathbb{R}^m$ is of class C^n if each $g_i: X \to \mathbb{R}$ is of class C^n.

A function like above is called C^∞ or <u>smooth</u> if it has continuous partials of all orders.

<u>Notes</u> ① This should be obvious, but if $f \in C^k$, then $f \in C^l$ $\forall l < k$.

② A continuous func is called C^0.

Something to think about

For $f: X \to \mathbb{R}^n \to \mathbb{R}$, suppose f is C^∞.

Then we know ① f has n 1^{st} partials

② f has n^2 2^{nd} partials (each 1^{st} partial is a diff. fnc of n-variables, so it has n-derivatives of its own.

③ f has n^k k^{th} partials, $\forall k \in \mathbb{N}$.

Now $Df(\vec{x})$ is a row matrix with n entries, each a ~~fnc~~ real-valued function on n vars.

Each of these fncs is differentiable, and if we view them as elements of a column, then

$$Df: X \subset \mathbb{R}^n \to \mathbb{R}^n.$$

And $D(Df) = D^2 f(\vec{x})$ will be a $n \times n$ matrix of functions, each real valued and diff.

And if each of the entries of $D^2f(\vec{x})$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is diff, each has n derivatives of its own, then one can think of $D^2f(\vec{x})$ as a function on X. What is its codomain?

What is its derivative? What kind of object is

$$D(D^2f)(\vec{x}) = D^3f(\vec{x})?$$

What about $D^k f(\vec{x})$, $k \in \mathbb{N}$?

These objects will play a role in the multivariable Taylor expansion of a func.

Think about this!