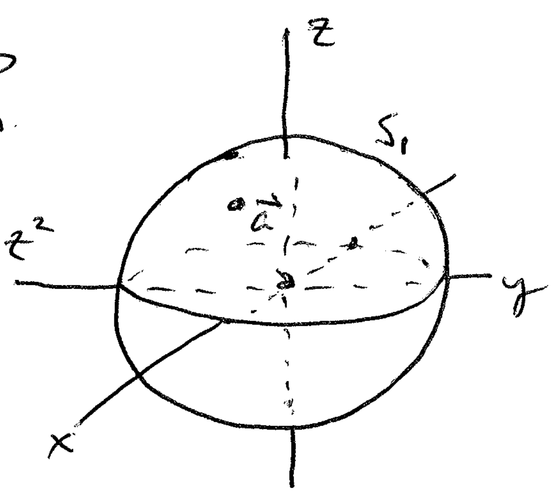
Section 2.6 (extra)

The Implicit Function Theorem

Let $F: X \subset \mathbb{R}^n \to \mathbb{R}$ be C^1, with $\vec{a} \in S_c$ where

$$S_c = \{ \vec{x} \in X \mid F(\vec{x}) = c \}.$$



ex. $F: \mathbb{R}^3 \to \mathbb{R}$ $\quad F(x,y,z) = x^2+y^2+z^2$

Let $\vec{a} \in S_1$.

Question: Is it possible to write S_1 as the graph of a function $z = f(x,y)$ in general? No! Why not?

Question: Is it possible "locally" near $\vec{a} \in S^1$?

Depends .... on what? If $\vec{a}$ is along the equator or not! Why?

Question: What information about F can be used to determine whether or not we can locally write a level set as the graph of a function?

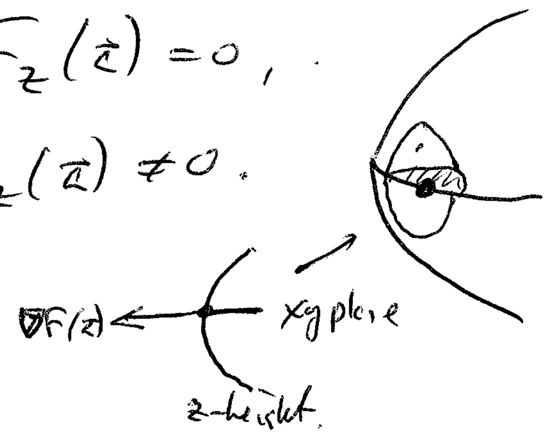
$36-18=18$

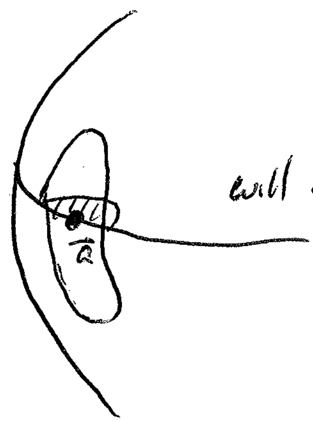
One idea? $\nabla F(\vec{z}) = \begin{bmatrix} F_x(\vec{z}) \\ F_y(\vec{z}) \\ F_z(\vec{z}) \end{bmatrix}$

What property does $\nabla F(\vec{z})$ have along the equator that it does not have elsewhere?

Along the equator, $F_z(\vec{z}) = 0$,

elsewhere, $F_z(\vec{z}) \neq 0$.





will contain pts in xy-plane with 2 values of z!

$F_z(\vec{z}) \neq 0$ is a ~~necessary~~ sufficient condition for ~~a loc~~ being able to locally write $F(x,y,z)=c$ near $\vec{z}$ in the level set as $z = f(x,y)$.

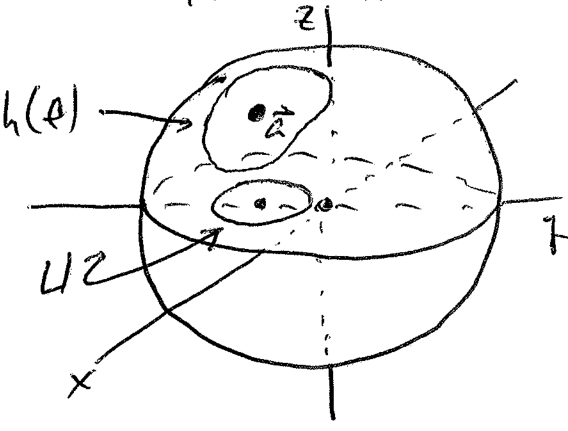
This works also in $\mathbb{R}^n$!

Thm 2.6.5

Let $F: X \subset \mathbb{R}^n \to \mathbb{R}$ be C^1 and $\vec{a} \in S_c$,
where $S_c = \{\vec{x} \in X \mid F(\vec{x}) = c\}$.

If $F_{x_n}(\vec{a}) \neq 0$, there exists a neighborhood U of $(a_1, a_2, \cdots, a_{n-1}) \in \mathbb{R}^{n-1}$, a nbhd V of $a_n \in \mathbb{R}$, and a function $f: U \subset \mathbb{R}^{n-1} \to V$ of class C^1 $\ni$

when $(x_1, \cdots, x_{n-1}) \in U$ and $x_n \in V$ satisfy $F(x_1, \cdots, x_n) = c$, then $x_n = f(x_1, \cdots, x_{n-1})$.



ex. $F(x,y,z) = 2xy^2 + xyz - 2z^2$, $\vec{a} = (2,-3,3)$. Can we write $z = f(x,y)$ near $\vec{a}$?

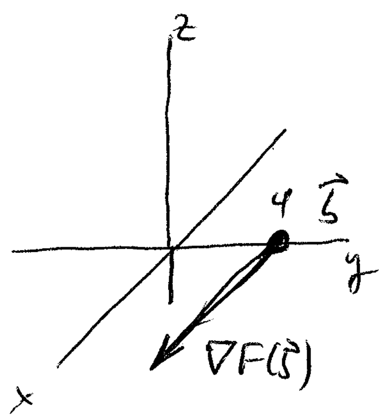
A: This is an easy yes, since $F_z(x,y,z) = xy - 4z$ and $F_z(\vec{a}) = xy - 4z\Big|_{(2,-3,3)} = -6 - 12 = -18 \neq 0$.

Since the z-component of $\nabla F(\vec{a})$ is not 0, it remains non-zero "near" $\vec{a}$ (in an open neighborhood). Thus the level surface is not "vertical" near $\vec{a}$.

At $\vec{\zeta} = (0, 4, 0)$, we have $\nabla F(\vec{\zeta}) = \begin{bmatrix} 2y^2 + yz \\ 4xy + xz \\ xy - 4z \end{bmatrix}\Bigg|_{\substack{x=0 \\ y=4 \\ z=0}}$

$$\nabla F(\vec{\zeta}) = \begin{bmatrix} 32 \\ 0 \\ 0 \end{bmatrix}$$

Look at the mathematics.



Note: Since ~~$\nabla F(\vec{\zeta})$~~ $F_x(\vec{\zeta}) \neq 0$, one could locally write the 0-level set of F as $x = f(y, z)$.

Here is a general version of the Implicit Function Theorem for vector valued funcs.

But for now, we will move on:

ex.

Back to example:

Find the tangent space to $F(x,y,z)=0$ at $\vec{a}=(2,-3,3)$, $\vec{b}=(0,4,0)$.

i) For $\vec{a}=(2,-3,3)$, $\nabla\vec{F}(\vec{a}) = \begin{bmatrix}2y^2+yz\\ 4xy+xz\\ xy-4z\end{bmatrix}\Bigg|_{(2,-3,3)} = \begin{bmatrix}9\\ -18\\ -18\end{bmatrix}$

Hence eqn for tangent plane is

$$\nabla\vec{F}(\vec{a})\cdot(\vec{x}-\vec{a}) = 0,$$

$$9(x-2) - 18(y+3) - 18(z-3) = 0, \text{ or}$$

$$18z = \underbrace{54 - 18y - 54}_{0} + 9x - 18$$

$$z = -1 + \tfrac{1}{2}x - y.$$

Note: $\vec{a} \in$ tangent plane. See drawing.

ii) For $\vec{b}=(0,4,0)$, $\nabla\vec{F}(\vec{b}) = \begin{bmatrix}16\\ 0\\ 0\end{bmatrix}$, so

$$\nabla\vec{F}(\vec{b})\cdot(\vec{x}-\vec{b}) = 0 = 16(x-0)+0(y-4)+0(z-0)=0$$

This is the $x=0$ plane (the yz-plane in $\mathbb{R}^3$).

Here $\vec{b}$ is in this plane and see figure.

The Inverse Function Thm

Suppose $\vec{f}: X \to \mathbb{R}^n \to \mathbb{R}^n$, $\vec{a} \in X$ open, $\vec{f} \in C^1$.

if det $D\vec{f}(\vec{a}) \neq 0$, then $\exists\, U(\vec{a}) \subset X$

an open nbhd, where ① $\vec{f}|_U$ is 1-1

② $\vec{f}(U) = V$ open in $\mathbb{R}^n$,

and ③ a uniquely defined inverse function

$\vec{g}: V \to U$, $g \in C^1$ where

$$\vec{g} \circ \vec{f} = Id_U \quad \text{and} \quad \vec{f} \circ \vec{g} = Id_V$$

(f and g are inverses of each other).

Notes ① Given $\vec{f}: X \to \mathbb{R}^n$, $\vec{f}(\vec{x}) = \begin{bmatrix} f_1(x_1 \cdots x_n) \\ \vdots \\ f_n(x_1 \cdots x_n) \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \vec{y}$.

is a system of n-eqns in n unknowns.

Question: Can we solve this system for $\vec{y}$, so that $\vec{y} = \vec{f}^{-1}(\vec{x})$ near a pt $\vec{a}$.

Answer: Yes if at $\vec{a}$, det $D\vec{f}(\vec{a}) \neq 0$.

(2) If $\vec{f}:\mathbb{R}^n \to \mathbb{R}^n$ is linear, then $\vec{f}(\vec{x}) = A_{n\times n}\vec{x} = \vec{y}$

Question: Is it possible to find a new matrix A^{-1} so that $\vec{x} = A^{-1}\vec{y}$?

A: Yes if $\det A \neq 0$.

<u>The Inverse Function Thm is the nonlinear version of this!</u>

ex. Can you solve $u = xy$, $v = x - y$ for x and y near the pt $(x,y) = (1,1)$?
How about near $(x,y) = (-1,1)$?
If so, invert the system.

Let $\vec{f}(x,y) = (u,v)$, $\vec{f}(x,y) = (xy, x-y) : \mathbb{R}^2 \to \mathbb{R}^2$

<u>Strategy</u>: Calculate the derivative matrix at $\vec{a} = (1,1)$ and $\vec{b} = (-1,1)$. If nonzero determinate at either, invert the system.

<u>Soln</u> $D\vec{f}(\vec{x}) = \begin{bmatrix} y & x \\ 1 & -1 \end{bmatrix}$ and $D\vec{f}(\vec{a}) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\det D\vec{f}(\vec{a}) = -2 \neq 0$

$D\vec{f}(\vec{b}) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $\det D\vec{f}(\vec{b}) = 0$.

Invert the system:

Since $v = x - y \Rightarrow x = v + y$, and then

$u = xy = (v+y)y = vy + y^2$. Write this

as $y^2 + vy - u = 0$ and solve for y:

$$y = \frac{-v \pm \sqrt{v^2+4u}}{2}.$$ Which branch?

Since when $(x,y) = (1,1)$, $(u,v) = (1,0)$, plug in

$u=1$, $v=0$ and choose branch so that $y=1$.

$\Longrightarrow$ we need the plus sign, so

$$y = \frac{-v + \sqrt{v^2+4u}}{2}.$$

And $x = v + y = v + \left(\frac{-v}{2}\right) + \frac{\sqrt{v^2+4u}}{2} = \frac{v+\sqrt{v^2+4u}}{2}.$

$$x = \frac{v + \sqrt{v^2+4u}}{2}.$$

Note: At $\vec{b}$, $x=-1$, $y=1$, $u=-1$, $v=-2$.

Can you see why x and y are <u>not</u> functions of u,v near $u=-1$, $v=-2$?