Sections 3.1 and 3.2.

We will skip Section 2.7 on Newton's Method
Also Kepler in Section 3.1 and curvature in 3.2.

Chapter 3 is a detailed look at 2 powerful tools for understanding structure in vector calculus and beyond: Both are vector-valued functions:

(a) Curves (paths) in $\mathbb{R}^n$
(b) Vector fields in $\mathbb{R}^n$

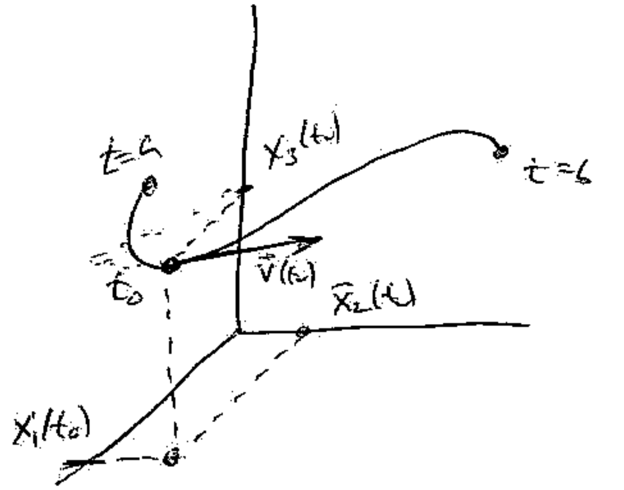
<u>Def</u> A curve or path in $\mathbb{R}^n$ is a <u>continuous</u> function $\vec{x}: I \to \mathbb{R}^n$, $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ on an interval (any) $I \subset \mathbb{R}$.

If $\vec{x}$ is differentiable as a function, then its derivative is also a vector (a $n \times 1$-matrix)

$$\frac{d}{dt}\vec{x}(t) = \vec{x}\,'(t) = \begin{bmatrix} \frac{dx_1}{dt}(t) \\ \vdots \\ \frac{dx_n}{dt}(t) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

which we sometimes call $\vec{v}(t) = \vec{x}\,'(t)$ the velocity

At a pt $t_0 \in I$, $\vec{v}(t_0)$ is represented by a vector, based at $\vec{x}(t_0) = \begin{bmatrix} x_1(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix}$ and tangent to $\vec{x}(t)$.

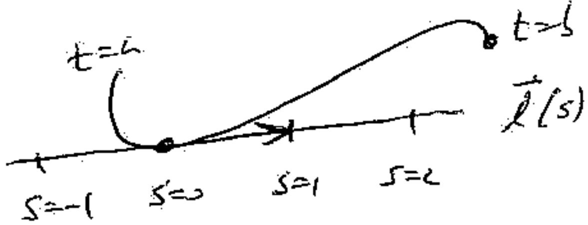


As long as $\vec{v}(t_0) \neq \vec{0}$, this vector defines a unique tangent line to $\vec{x}(t)$ at t_0 parameterized as

$$\vec{\ell}(s) = \vec{x}(t_0) + s\,\vec{v}(t_0), \quad \text{or}$$

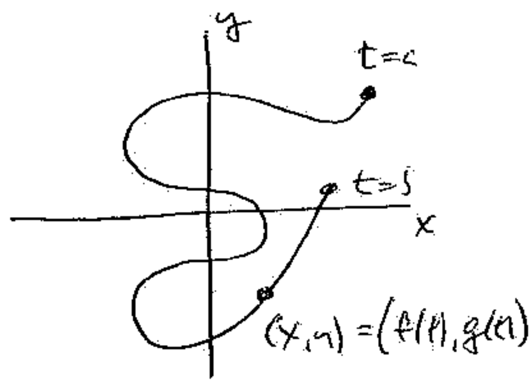
$$\vec{\ell}(t) = \vec{x}(t_0) + (t-t_0)\,\vec{v}(t_0), \quad s = t - t_0$$

and a $\vec{\ell} = \operatorname{span}\{\vec{v}(t_0)\}$.



Here, the speed of $\vec{x}(t)$ at $t=t_0$ is $\|\vec{v}(t_0)\|$, the length of the tangent vector. Think of a bead traveling along the wire.

You have seen this all before! Calculus II



$x = f(t)$, $y = g(t)$, for $t \in I \subset \mathbb{R}$, defines a parametric curve in $\mathbb{R}^2$

- if $f, g \in C^1$, then $\frac{dx}{dt} = f'(t)$

$$\frac{dy}{dt} = g'(t)$$

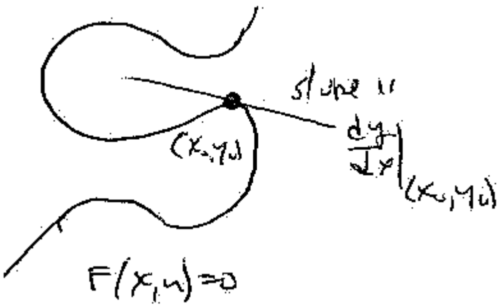
and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ defines the tangent line in $\mathbb{R}^2$ to curve at (x_0, y_0):

$$y = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} (x - x_0) + y_0, \qquad x_0 = f(t_0),\ y_0 = g(t_0).$$

This was constructed to study curves that are defined only implicitly (not representable as a function explicitly).

If a curve is defined as $F(x,y)=0$, then

① we can still calculate $\frac{dy}{dx}$ implicitly, or



② we can parameterize the curve via $x=f(t)$, $y=g(t)$, so that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

<u>In the first way</u>: Assume $y = y(x)$ is an implicit fnc of x

Then $F(x,y)=0$ is $F(x,y(x))=0$ is only a fnc of x

and $\frac{d}{dx}F(x,y(x)) = \frac{\partial}{\partial x}F(x,y) + \frac{\partial}{\partial y}F(x,y)\frac{dy}{dx} = 0$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{-F_x(x,y)}{F_y(x,y)}$$

<u>In the second way</u>: Both x,y are functions of t, so

$F(x,y)=0$ is $F(x(t),y(t))=0$ is only a fnc of t

Here $\frac{d}{dt}F(x(t),y(t)) = \frac{\partial}{\partial x}F(x,y)\frac{dx}{dt} + \frac{\partial}{\partial y}F(x,y)\frac{dy}{dt} =$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-F_x(x,y)}{F_y(x,y)} \qquad \underline{\text{Same!}}$$

Thinking of a curve as a function allows us all of the tools of calculus to study the geometry of curves:

① Define higher derivatives like acceleration, jerk, etc.

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d^2}{dt^2}\vec{x}(t).$$

② We can recover quantities like distance via integrating velocity:

$$\vec{x}(t) = \int \vec{v}(t)\,dt.$$

Keep in mind the that integrating a vector is integrating each component; the constant of integration is a vector.

③ Rules behave well with regard to curves:

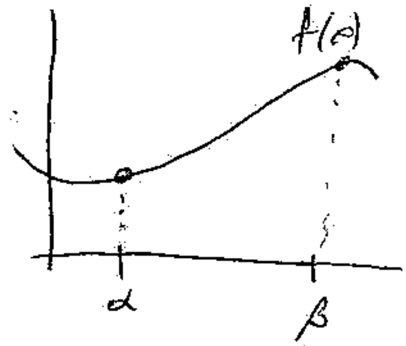
$$\frac{d}{dt}(\vec{x}(t)\cdot\vec{y}(t)) = \frac{d\vec{x}}{dt}\cdot\vec{y} + \vec{x}\cdot\frac{d\vec{y}}{dt} \quad \text{etc.}$$

④ Facilitates geometric study:

<u>Example</u>: if $\vec{x}(t) \subset \mathbb{R}^n$ is a diff. curve, with $\|\vec{x}(t)\| = c > 0 \ \forall t$, then $\vec{x}(t)\cdot\vec{x}'(t) = 0 \ \forall t$.

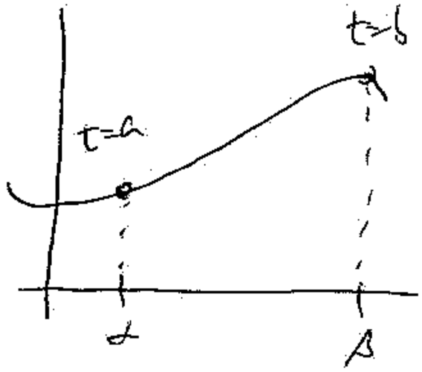
Exercise: Prove this.

Recall from Calculus II, for $f: [\alpha, \beta] \to \mathbb{R}$, the length of graph$(f) \subset \mathbb{R}^2$ on $[\alpha, \beta]$ is



$$L = \int_\alpha^\beta \sqrt{1 + (f'(x))^2}\, dx,$$

or if the curve is *parametric*

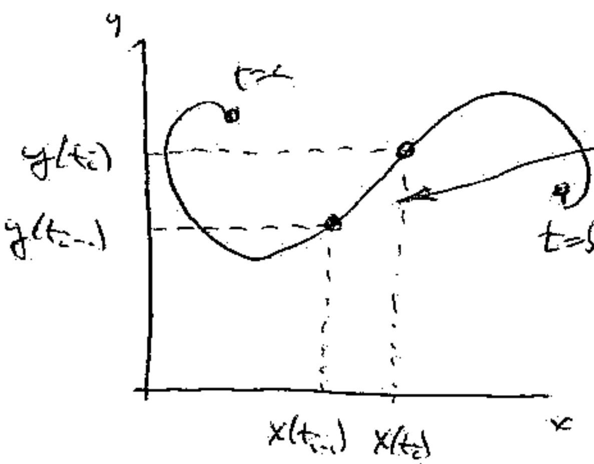


$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

exercise: show these are equivalent.

This last formula is tailor-made for us:

Let $\vec{x}: [a, b] \to \mathbb{R}^2$, $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.



The approximate length on $\Delta t_i = [t_{i-1}, t_i]$

$$x(t_i) - x(t_{i-1}) = \Delta x_i = x'(t_i^*)\, \Delta t_i$$
$$y(t_i) - y(t_{i-1}) = \Delta y_i = y'(t_i^{**})\, \Delta t_i$$

by MVT for some $t_i^* \in [t_{i-1}, t_i]$, and t_i^{**}

So the approx. length of the curve is

$$\text{approx } L = \sum_{i=1}^n \sqrt{(x'(t_i^*))^2 + (y'(t_i^{**}))^2}\, \Delta t_i$$

and the actual length is in the limit as all $\Delta t_i \to 0$

$$L = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{(x'(t_i^*))^2 + (y'(t_i^{**}))^2}\, \Delta t_i$$

$$= \int_a^b \underbrace{\sqrt{(x'(t))^2 + (y'(t))^2}}_{\text{this is just the speed } \|\vec{x}'(t)\|}\, dt$$

and works in $\mathbb{R}^n$:

<u>Def</u> The length of a C^1-parameterized curve in $\mathbb{R}^n$ $\vec{x} : [a,b] \longrightarrow \mathbb{R}^n$ is $L(\vec{x}) = \int_a^b \|\vec{x}'(t)\|\, dt$

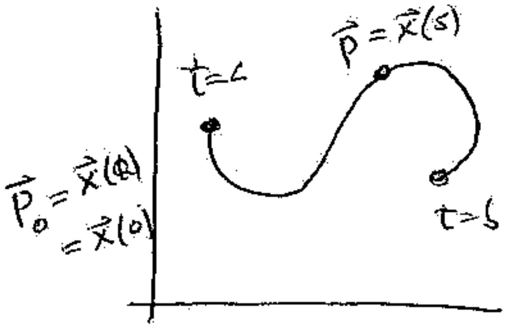
<u>Notes</u> ① Integrate speed wrt time to recover distance!

② Even if only piecewise C^1 this is okay. (sum the integrals on each piece).

③ Seems to depend critically on parameterization. But it does not! To see this, reparameterize and reintegrate!

Or parameterize intrinsically (use length itself as a parameter!).

Let $\vec{x}:[a,b]\to\mathbb{R}^n$ be a path w/ nonzero velocity everywhere. Let $\vec{p}=\vec{x}(a)$, and then for any



other pt $\vec{P}_0\in\vec{x}(I)$, $\vec{p}=\vec{x}(s)$, where

$$s(t)=\int_a^t \|\vec{x}\,'(\tau)\|\,d\tau \leftarrow \text{dummy var. in a def. integral.}$$

- Since $\vec{x}\,'(t)\neq\vec{0}$, then, its length is always positive, so $s(t)$ is always an increasing function: It is invertible, and we can reparameterize

$$\vec{x}(t)=\vec{x}(t(s)) \quad \text{as a fnc of } s.$$

- In practice, $t(s)$ may be difficult to find, but the total length of the curve is

$$s(b)=\int_a^b \|\vec{x}\,'(\tau)\|\,d\tau=\int_a^b \|\vec{x}\,'(t)\|\,dt = \text{length of curve in } t \text{ parameter}$$

Reparameterization does not change length.

- $s(t)$ is C^1 when $\vec{x}$ is, and

$$s'(t)=\frac{ds}{dt}=\frac{d}{dt}\left(\int_a^t \|\vec{x}\,'(\tau)\|\,d\tau\right)=\|\vec{x}\,'(t)\|$$

So under this reparameterization, the derivative is just the speed of the curve at the old value for t.

So we can use this to calculate the tangent vector to the curve in the new parameter:

Write $\vec{x}(t) = \vec{x}(s(t))$. Then differentiate:

$$\vec{x}\,'(t) = \frac{d}{dt}\vec{x}(s(t)) = \vec{x}\,'(s)\cdot s'(t) = \vec{x}\,'(s)\,\|\vec{x}\,'(t)\|$$

So $$\vec{x}\,'(s) = \frac{\vec{x}\,'(t)}{\|\vec{x}\,'(t)\|}.$$

Conclusions: (1) In the new parameter, the arclength traverses the curve at unit speed always.

(2) $\vec{x}\,'(s)$ is just the <u>normalization</u> of the tangent vector at the same pt as $\vec{x}(t)$.

<u>Def</u> For a C^1-path $\vec{x}:[a,b]\to\mathbb{R}^n$, the <u>unit tangent vector</u> to $\vec{x}$ at $t=t_0$ is

$$T = \frac{\vec{x}\,'(t)}{\|\vec{x}\,'(t)\|}$$

and is just the normalized velocity.

This will be an important concept later on.