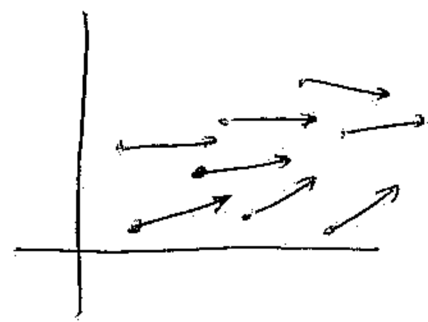
Section 3.3 and 3.4

<u>Def.</u> A vector field on $\mathbb{R}^n$ is a map

$$F : X \subset \mathbb{R}^n \to \mathbb{R}^n$$

and is the assignment of a vector $F(\vec{x})$ based at each $\vec{x} \in X$



Examples include

ⓐ Force fields in physics,

ⓑ Slope fields in differential equations.

ⓒ Fluid (air) flow in climate models.

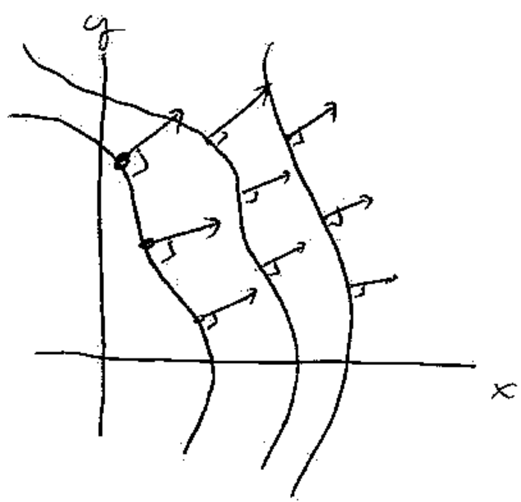
The vector field is called C^n when F is C^n:

- means vectors vary in both size and direction continuously, or differentiably, or etc...

<u>Def</u> A vector field is called a <u>gradient field</u> on $\mathbb{R}^n$ if F is the gradient of a real-valued function $f : X \subset \mathbb{R}^n \to \mathbb{R}$:

Let $f: X \subset \mathbb{R}^n \to \mathbb{R}$ be C^1. Then

$\nabla f: X \subset \mathbb{R}^n \to \mathbb{R}^n$. Here, this can be interpreted as a vector field on X, called the gradient field of f.



- Here, f is called a <u>potential function</u> for $F(\vec{x}) = \nabla f(\vec{x})$, a vector field on X.

- Recall for $f: \mathbb{R}^2 \to \mathbb{R}$, its level sets $\{f(\vec{x}) = c\}$, $c \in \mathbb{R}$ were curves in $\mathbb{R}^2$. For f a potential function, we have:

  ⓐ The level curves are equipotential <u>sets</u>,

  ⓑ The gradient field along these sets always is orthogonal to (the tangent lines of) these sets

  ⓒ The gradient field always points in the direction of the most rapid increase of f at that pt.

ⓓ In contrast to a vector field, a real-valued fnc is sometimes called a scalar-field. The gradient takes a potential (scalar) field to a gradient (vector) field.

Example

Given a scalar function $f: X \subset \mathbb{R}^n \to \mathbb{R}$,
finding its gradient field is easy: take derivatives and form a vector.

Given a gradient field, tho, how to find its potential?

In Example 5, pg 231, Author tells you the gradient field $\vec{F} = \begin{bmatrix} 3x^2z + y^2 \\ 2xy \\ x^3 - 2z \end{bmatrix}$ has potential $f(x,y,z) = x^3z + xy^2 - z^2$, and "leave[s] it to you ..." to figure out how to find it?

Idea: Undifferentiate!

① $\frac{\partial f}{\partial y} = 2xy$. Hence $f(x,y,z) = xy^2 + h(x,z)$ why?

② $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy^2 + h(x,z)) = y^2 + \frac{\partial h}{\partial x}(x,z) = 3x^2z + y^2$

$\implies h(x,z) = x^3z + g(z)$

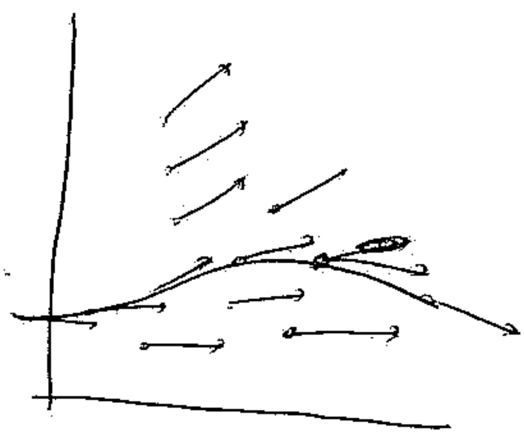
③ $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(xy^2 + x^3z + g(z)\right) = x^3 + g'(z) = x^3 - 2z.$

$\implies g'(z) = -2z,\ g(z) = z^2.$

$\implies f(x,y,z) = xy^2 + x^3z - z^2.$ ∎

Definition A flow line (trajectory) of a vector field $\vec{F}: X \subset \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable curve $\vec{x}: I \to \mathbb{R}^n$ such that

$$\vec{x}'(t) = \vec{F}(\vec{x}(t)) \qquad \forall t \in I$$



- Here the velocity vector of the path at every $t \in I$ equals the vector field at that pt.

- Finding such a path, given a vector field, is precisely the subject of differentiable equations.

ex Is the path $\vec{x}(t) = \begin{bmatrix} e^{-t} + 2e^{2t} \\ -e^{-t} + e^{2t} \end{bmatrix}$ a flow line of the vector field $\vec{F}(\vec{x}) = \begin{bmatrix} x+2y \\ x \end{bmatrix}$?

Answer: Yes, since

$$\vec{x}'(t) = \begin{bmatrix} -e^{-t} + 4e^{2t} \\ e^{-t} + 2e^{2t} \end{bmatrix} = \begin{bmatrix} (e^{-t} + 2e^{2t}) + 2(-e^{-t} + e^{2t}) \\ (e^{-t} + 2e^{2t}) \end{bmatrix}$$

$$= \begin{bmatrix} x(t) + 2y(t) \\ x(t) \end{bmatrix} = F(\vec{x}(t)) \quad \blacksquare$$

Def. An *operator* is a mapping from one linear (vector) space to another.

- Sometimes, spaces of functions can be linear
  - any linear combination of functions is a function,
  - there is an additive identity (0-fnc
- So operators can take fncs to fncs.

Def. The *del operator* is the operator that takes a real-valued C^1-function $f: X \subset \mathbb{R}^n \to \mathbb{R}$

to its gradient vector field $\nabla f: X \subset \mathbb{R}^n \to \mathbb{R}^n$.

Notation: $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ in $\mathbb{R}^3$, or

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} = \vec{e}_1 \frac{\partial}{\partial x_1} + \cdots + \vec{e}_n \frac{\partial}{\partial x_n} \text{ in } \mathbb{R}^n$$

This notation is odd (but common) and implies

$$\nabla(\quad) = \sum_{i=1}^{n} \vec{e}_i \frac{\partial}{\partial x_i}(\quad) = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}$$

insert fnc here

interpreted as $\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}(f) = \begin{bmatrix} f_{x_1} \\ \vdots \\ f_{x_n} \end{bmatrix}$ for $f: X \subset \mathbb{R}^n \to \mathbb{R}$

<u>Def</u> For $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ a C^1 vector field, the <u>divergence</u> of $\vec{F}$, denoted $\operatorname{div}\vec{F}$, or $\nabla \cdot \vec{F}$ is the scalar function

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$$

$$= \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}$$

for $\vec{x} = (x_1, \dots, x_n) \in X$ and $\vec{F}(\vec{x}) = \begin{bmatrix} F_1(\vec{x}) \\ \vdots \\ F_n(\vec{x}) \end{bmatrix} \in \mathbb{R}^n$.

<u>Notes</u> ① Here $\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} \cdot \begin{bmatrix} F_1(x_1,\dots,x_n) \\ \vdots \\ F_n(x_1,\dots,x_n) \end{bmatrix}$

dot prod

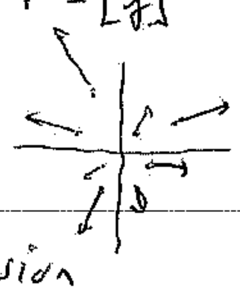
② We will prove this later, but divergence measures infinitesimal volume change in a vector field.

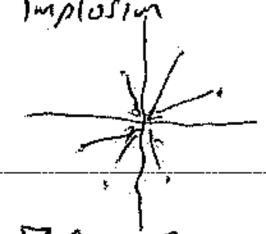
③ A vector field $\vec{F}$ whose $\nabla \cdot \vec{F} = 0$ is called <u>incompressible</u>.

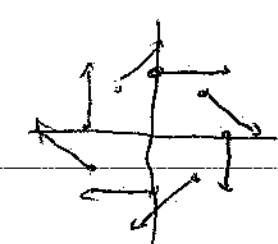
<u>exs</u>

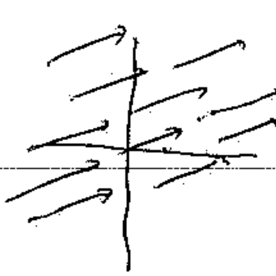
$\vec{F} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \vec{G} = \begin{bmatrix} -x \\ -y \end{bmatrix}$ $\quad \vec{H} = \begin{bmatrix} y \\ -x \end{bmatrix}$ $\quad \vec{J} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Implosion









explosion

$\nabla \cdot F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2 > 0$

$\nabla \cdot G = -2$.

$\nabla \cdot \vec{H} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) = 0$

$\nabla \cdot \vec{J} = 0$.

④ As an operator, ∇ can operate on fncs in different ways. VI

ⓐ Gradient of a scalar field: $\nabla f(\vec{x})$ ← scalar mult

ⓑ Divergence of a vector field: $\nabla \cdot \vec{F}$ ← dot prod

ⓒ Curl of a vector field in $\mathbb{R}^3$: $\nabla \times \vec{F}$ ← cross prod

<u>Def</u> For $\vec{F}: X \subset \mathbb{R}^3 \to \mathbb{R}^3$ a C^1 vector field in $\mathbb{R}^3$, the <u>curl</u> of $\vec{F}$, curl $\vec{F}$ or $\nabla \times \vec{F}$ is

$$\nabla \times \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

<u>Notes</u> ① Notice that ⓐ the gradient of a scalar field is a vector field.

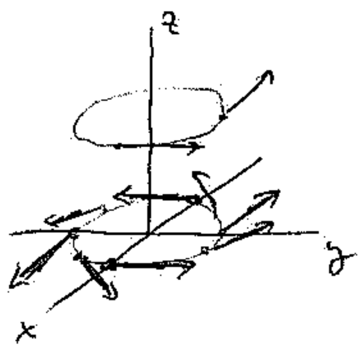
ⓑ the divergence of a vector field is a scalar field

ⓒ the curl of a vector field (in $\mathbb{R}^3$) is a <u>vector field</u>!

② We will prove this later, but the curl measures the infinitessimal twist in the vector field at each pt

③ If, for $\vec{F}$, we have $\nabla \times \vec{F} = \vec{0}$ everywhere, we say $\vec{F}$ is irrotational.

ex $\vec{F} = \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}$ rotates each xy-plane at $z = c$.



Here $\nabla \times \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}$

$$= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-x)\right)\vec{i} - \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y)\right)\vec{j} + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right)\vec{k}$$

$$= -2\vec{k}$$

Here $\nabla \times \vec{F}$ is a vector field that must be orthogonal to $\vec{F}$ since it is a cross product.

ex. Explosion/Implosion in $\mathbb{R}^3$: $\vec{C} = \begin{bmatrix} \pm x \\ \pm y \\ \pm z \end{bmatrix}$. Here $\nabla \times \vec{C} = \vec{0}$.

ex. Constant vector fields: $\vec{H} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\nabla \times \vec{H} = \vec{0}$.

④ Some beautiful facts:

ⓐ Curl of the gradient: $\nabla \times (\nabla f) = \vec{0}$

Gradient fields are irrotational.

ⓑ Divergence of the curl: $\nabla \cdot (\nabla \times \vec{F}) = 0$

The curl of any vector field is incompressible.