Section 4.2.

Like in Calculus I/II, local extrema (and global extrema) are very important to an understanding of functions.

Def $f: X \subset \mathbb{R}^n \to \mathbb{R}$, X open, has a local minimum at $\vec{a} \in X$ if $\exists$ a nbhd $U(\vec{a})$ $\ni$ $f(\vec{x}) \geq f(\vec{a})$ $\forall \vec{x} \in U$. [local max if inequality is reversed].

Notes ① A local min (max) is global if $U = X$.

② If $f \in C^1$, local extrema have a special quality:

Thm Given X open and $f: X \subset \mathbb{R}^n \to \mathbb{R}$ a C^1 func, if f has a local extremum at $\vec{a}$, $\Rightarrow$ $Df(\vec{a}) = 0_{1 \times n}$

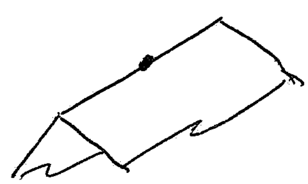
③ Proof shows that the directional derivative (a calc I derivative) also ~~has~~ reflects a local extremum at $\bar{a}$ in <u>every</u> direction! (In particular, in the coord. directions as elements of $Df(\bar{a})$).

<u>Def.</u> A pt $\bar{a} \in X$ open is a critical pt of $f: X \subset \mathbb{R}^n \to \mathbb{R}$ if ① $Df(\bar{a}) = 0_{1\times n}$, or ② $Df(\bar{a})$ is undefined.

<u>Note</u>: Just like in Calc I/II, extrema happen at critical pts but not all critical pts need be extrema!

all qualify as critical:

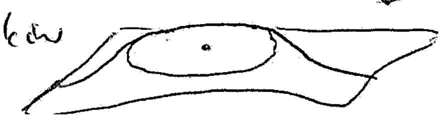
- Ridges



- Mountain tops and valleys.

- Saddles

- Mesas
  Floodplains



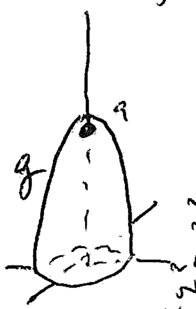
But extrema do not have to exist at all!

<u>ex.</u> Let $f(x,y) = x^2+y^2$ on $\mathbb{R}^2 - (0,0)$.

Does f have extrema?

How to detect an extremum from a critical pt?

– it's all about structure.



$g(x,y,z) = 9 - x^2 - y^2$

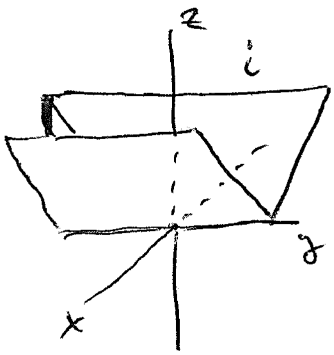
① Look for extreme behavior by simply testing function values "near" the critical pt. (0th-derivative test?).

This method becomes more important when $Df(\vec{a})$ is not defined at a critical $\vec{a}$.

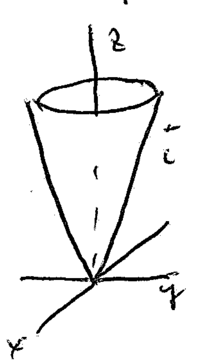
$h(x,y) = |x|$

$i(x,y) = \sqrt{x^2+y^2}$

② If f is diff at a critical pt $\vec{a}$, then $Df(\vec{a}) = 0_{1\times n}$. Hence every directional derivative is 0 at $\vec{a}$.

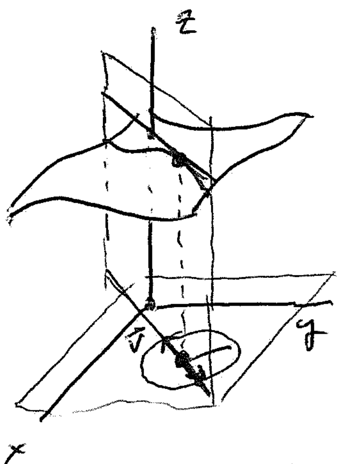


Recall directional derivatives were defined via vertical slices through the graph of f along directions in the domain.



(2) cont'd.

If one follows the curve in that slice and sees the derivative go from negative (before $\vec{a}$) to 0 (at $\vec{a}$) to positive (after $\vec{a}$), AND this happens in EVERY direction, then you have a local minimum at $\vec{a}$.



~~If at least, then~~ (Or, at least, the directional derivative remains at 0 for some direction) [1st derivative test?].

(3) Or, if within each slice, f restricted to the slice is concave up (down) then $\vec{a}$ is locally extreme (in that dir!!

(Really, all that is needed is that there is no mixed concavity).

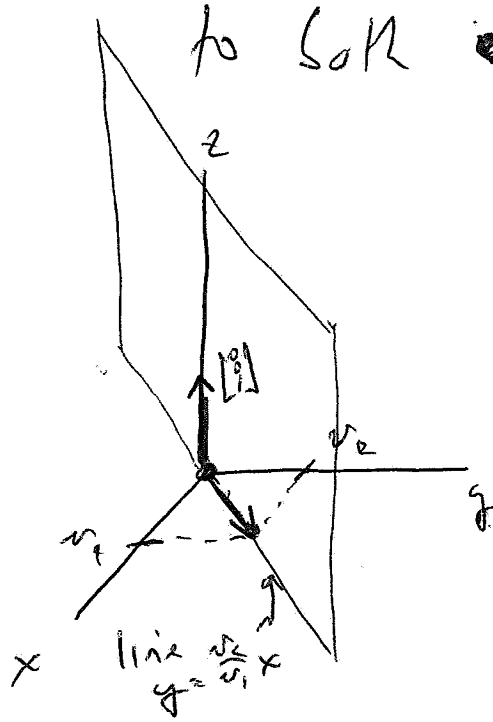
If locally extreme in every direction, then you are assured locally extreme.

Ex. Let $f(x,y) = x^2+y^2$. Here, graph(f) is the parabolic bowl in $\mathbb{R}^3$. $f \in C^1$ everywhere, and $Df(\vec{x}) = [2x \;\; 2y] = [0 \;\; 0]$ only when $x = y = 0$.

So here $D_{\vec{v}}f\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = 0 \quad \forall \vec{v} \in \mathbb{R}^2$

$$= Df(\vec{0})\vec{v} = [0 \;\; 0]\begin{bmatrix}v_1\\v_2\end{bmatrix} = 0$$

Choose a direction $\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix} \in X = \mathbb{R}^2 \subset \mathbb{R}^3$, the domain. Then the vertical plane containing $\vec{v}$ is defined by the vector orthogonal to both $\begin{bmatrix}v_1\\v_2\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$, and is



$$\begin{bmatrix}v_1\\v_2\\0\end{bmatrix} \times \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ v_1 & v_2 & 0\\ 0 & 0 & 1\end{vmatrix} = \begin{bmatrix}v_2\\-v_1\\0\end{bmatrix}, \text{ or}$$

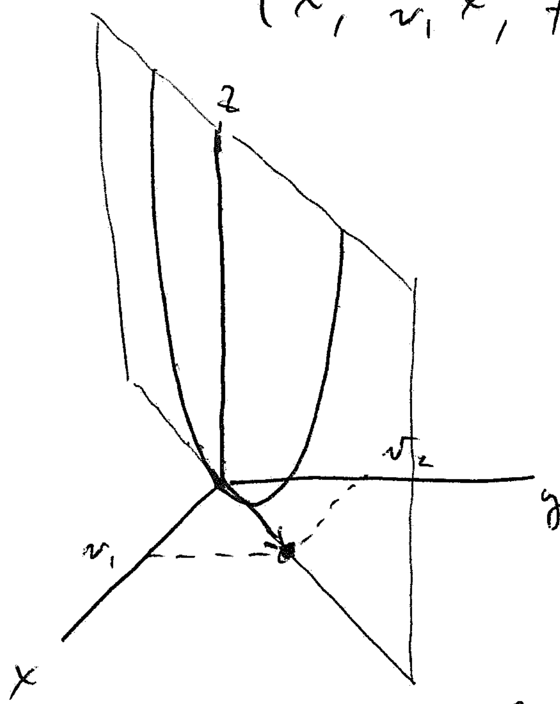
$$v_2 x - v_1 y = 0$$

Eqn of vertical slice in $\vec{v}$ direction.

In the domain, this is the line $y = \frac{v_2}{v_1}x$, at least when $v_1 \neq 0$.

The graph of this line is the set of pts

$$\left(x, \tfrac{v_2}{v_1}x, f(x, \tfrac{v_2}{v_1}x)\right) = \left(x, \tfrac{v_2}{v_1}x, x^2\left(1+\tfrac{v_2^2}{v_1^2}\right)\right) \in \mathbb{R}^3.$$



Let $f_{\vec{v}} = f\Big|_{y=\frac{v_2}{v_1}x} : \mathbb{R} \to \mathbb{R}$

where $f_{\vec{v}}(x) = x^2\left(1+\frac{v_2^2}{v_1^2}\right)$

Here, of course, $f'_{\vec{v}}(0) = 2x\left(1+\frac{v_2^2}{v_1^2}\right)\Big|_{x=0} = 0$

and $f''_{\vec{v}}(0) = 2\left(1+\frac{v_2^2}{v_1^2}\right) > 0$.

Hence $f_{\vec{v}}$ is concave up and, at least in this direction, according to Calc I 2nd derivative Test, 0 is a local minimum of $f|_{\vec{v}}$ (0 corresponds to the origin).

Note that this will remain true for <u>all</u> directions (even if $v_1 = 0$).

But for $f : X \subset \mathbb{R}^n \to \mathbb{R}$ a C^2-function, we already have access to 2nd derivative information for all directions: <u>The Hessian Matrix</u> $Hf(\bar{x}) = \left[f_{x_i x_j}(\bar{x}) \right]$.

Recall $D_{\vec{v}} f(\bar{x}) = Df(\bar{x})\vec{v}$. One can also show that $D^2_{\vec{v}} f(\bar{x}) = \vec{v}^T Hf(\bar{x}) \vec{v}$ is the 2nd directional derivative of f in the direction of $\vec{v}$.

It measures directly the concavity of the slice of graph of f at $\bar{x}$ and containing $\vec{v}$.

Hence if $D^2_{\vec{v}} f(\bar{x}) = \vec{v}^T Hf(\bar{x}) \vec{v} > 0$

$\forall \vec{v}$ in X based at $\bar{x}$, then one has a local min of f at $\bar{x}$.

① **Note:** For any $n \times n$ matrix $A_{n\times n}$ one can construct a quadratic form: a ~~function~~ real valued function $Q: \mathbb{R}^n \to \mathbb{R}$ $\ni$

In one var $Q(x) = ax^2$

In 2 vars $Q(x) = ax^2 + bxy + cy^2$

$$Q(\vec{x}) = \vec{x}^T A \vec{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

② Quadratic forms are invariant under conjugation. Hence we can always take A to be symmetric. ($A^T = A$), or ($a_{ij} = a_{ji}$).

③ Hermitians are always symm. ~~Hence~~

④ $Q(x)$ is called positive def. if

$$Q(\vec{x}) > 0 \quad \forall \vec{x} \neq \vec{0}$$

Negative def if

$$Q(\vec{x}) < 0 \quad \forall \vec{x} \neq \vec{0}.$$

~~Hello~~
IX

Thm 4.2.3 For $X \subset \mathbb{R}^n$ open, let $f: X \to \mathbb{R}$ be C^2 with a critical pt $\vec{a} \in X$.

1. If $Hf(\vec{a})$ is pos. def., then f has a local min at $\vec{a}$.
2. If $Hf(\vec{a})$ is negative def, then f has a local max at $\vec{a}$.
3. If $\det Hf(\vec{a}) \neq 0$ and neither pos nor neg def then $\vec{a}$ is a saddle.

There is a ~~mech~~ mechanical process for det. pos/neg definiteness and it is all linear algebra.

In essence, it involves testing the determinants of principal minors to see if $Hf(\vec{a})$ is positive def, neg def or neither.

Positive definiteness of a Quadratic form

$$Q(\vec{x}) = \vec{x}^T A_{n\times n} \vec{x}.$$

- ~~Here~~ Define the k^{th} principal minor of A to be $A_k = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$

This is a $k \times k$ matrix, and A has n of them

$$A_1 = a_{11}, \quad A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \cdots \quad A_n = A.$$

non degenerate

- Here, $Q(\vec{x})$ (and hence A) is positive def if $\det(A_k) > 0 \quad \forall\, k = 1, \ldots, n.$
- $Q(\vec{x})$ (and hence A) is negative def if $\det(A_k) < 0$ for k-odd, and $\det(A_k) > 0$ for k-even.
- If neither and $\det A_k \neq 0$, ~~[illegible]~~ called indefinite
- If $\det A_k = 0$, then called degenerate.

Lastly, the Extreme Value Thm from Calc I has a counterpart in Calc III.

Def. ① A set X is closed if it contains all of its boundary pts.

② A set $X \subset \mathbb{R}^n$ is bounded if $\exists M > 0$ $\ni \|\vec{x}\| < M \ \forall \vec{x} \in X$.

③ A set $X \subset \mathbb{R}^n$ is <u>compact</u> if it is both <u>closed</u> and <u>bounded.</u>

<u>Thm</u> [EVT] If $X \subset \mathbb{R}^n$ is compact and $f: X \to \mathbb{R}$ continuous, then f has a global max and global min on X.

<u>Note</u>: Like in calc I, even for X open or unbounded, f may have global extrema. But maybe not. And it must if X compact and f at least C^0.