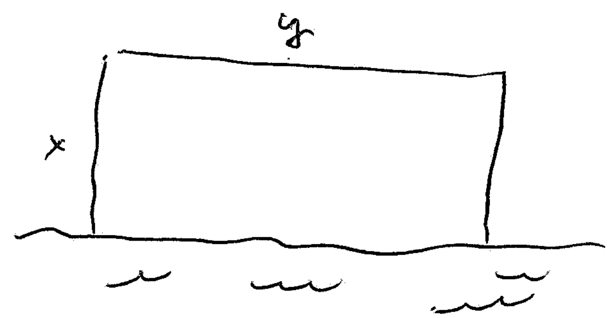
Section 4.3

Recall optimization in Calculus I:

ex. Using 1800 linear feet of fence, construct a rectangular yard along a straight river with the largest area possible:

Idea: Maximize $A = xy$
Subject to
$1800 = 2x + y$.



Here A is the objective fnc.
$1800 = 2x + y$ is the constraint

The constraint facilitates calculation by
① allowing us to change the objective into a fnc of one variable.
② allows us to use single var. calculus techniques to help solve.

Here, since $1800 = 2\cancel{x} + y$, $y = 1800 - 2x$, so

$$A = xy = x(1800 - 2x) = 1800x - 2x^2.$$

Note the shape of (the graph of) $A(x)$,

a parabola opening down with a max

at its vertex.

Calculus: Look for critical pts:

$A(x)$ is C^1, so only at places where $A' = 0$

$$A'(x) = 1800 - 4x = 0$$

when $x = 450$.

Here $A''(450) = -4 < 0$ hence by 2nd der test,

$x = 450$ is a local max.

Solution: Dimension of yard are x = width = 450

and y = length = 900, with area

$$A = xy = 450\,(900) = \ldots$$

Notes ① Here, it is implicit that $x>0$, $y>0$

so "region" in xy-plane is the

open first quadrant.

~~② ∃! critical pt of A(x)~~

② Also implicit: $x < \cancel{1800}\ 900$, $y < 1800$.

③ $\exists!$ critical pt of $A(x)$ on $(0, 900)$

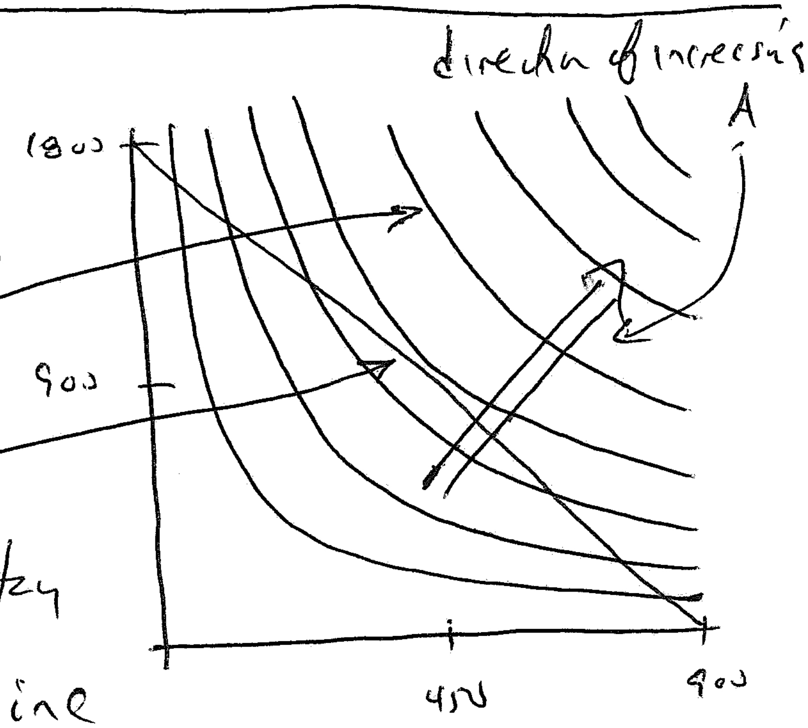
and since "edges" go to 0 and $A(x)>0$

on interval, of course it is a max.

Different viewpoint:

$A = xy$ level sets

Constraint



If we are forced to stay

on the constraint line

and look for the largest value of A along it.

where will we find it?

Here, we leave both functions as fncs of both x and y and look for a geometric ~~reason~~ way to find an extremum. of the objective given the constraint.

One can see (maybe look?) that along the constraint line, we are cutting through level sets of A for a while then we go tangent and then we again cut through level sets.

What is happening to the values of A then?

This new view allows us to generalize:

Optimize $f: X \subset \mathbb{R}^n \to \mathbb{R}$ (X open)

subject to $g: X \to \mathbb{R}$ where $g(\vec{x}) = c$.

We look for extrema of f while "stuck" on the c-level set of g.

Note: We could simply solve $g(\vec{x}) = c$ for one of its variables in terms of the other, ~~but~~ and then "lower" the number of variables of f by 1.

But sometimes this is not possible!

ex. Max $f(x,y,z) = x^2 + 3y^2 + y^2 z^4$

subject to $g(x,y,z) = e^{xy} - xyz + \cos\left(\frac{xy}{z}\right) = 2$.

Instead, appeal to the geometry!

Thm 4.3.1 For $X \subset \mathbb{R}^n$ open, $f, g : X \to \mathbb{R}$ C^1-functions, let

$$S = \{\vec{x} \in X \mid g(\vec{x}) = c\}$$

be the c-level set of g. Then

If $f|_S$ has an extremum at $\vec{x}_0 \in S$, where $\nabla g(\vec{x}_0) \neq \vec{0}$, then $\exists \lambda \in \mathbb{R} \ni$

$$\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0)$$

Notes ① Extrema of f will happen at places where ∇f is a multiple of ∇g, as vectors.

② $\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0)$ is a set of n-equations (nonlinear) in $n+1$-unknowns (all $\vec{x}$, and λ).

$\Rightarrow$ Lots of solns!!!

(3) But add in the constraint, and

$$f_{x_1}(\bar{x}) = \lambda g_{x_1}(\bar{x})$$
$$\vdots$$
$$f_{x_n}(\bar{x}) = \lambda g_{x_n}(\bar{x})$$
$$g(\bar{x}) = c$$

is a set of $(n+1)$-eqns in $(n+1)$-unknowns (typically isolated solutions).

(4) λ is called a Lagrange Multiplier. Not usually important what its value is....

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ex. Identify all critical pts of

$$f(x,y) = 5x + 2y$$

subject to $g(x,y) = 5x^2 + 2y^2 = 14$.

Soln Here $\nabla f(\bar{x}) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $\nabla g(\bar{x}) = \begin{bmatrix} 10x \\ 4y \end{bmatrix}$

So system is

$$f_x(\bar{x}) = \lambda g_x(\bar{x})$$
$$f_y(\bar{x}) = \lambda g_y(\bar{x})$$
$$g(\bar{x}) = c$$

$$\boxed{\begin{aligned} &① \quad 5 = \lambda 10x \\ &② \quad 2 = \lambda 4y \\ &③ \quad 5x^2 + 2y^2 = 14 \end{aligned}}$$

Mess with this: By ①, ② $x = \frac{1}{2\lambda} = y$.

so by ③ $\frac{5}{4\lambda^2} + \frac{2}{4\lambda^2} = 14 \Rightarrow \frac{1}{4\lambda^2} = 2$

or $\lambda^2 = \frac{1}{8} \Rightarrow \lambda = \pm \frac{1}{2\sqrt{2}}$

Hence $(x, y) = (\sqrt{2}, \sqrt{2})$, or
$(x, y) = (-\sqrt{2}, -\sqrt{2})$. See McClenachan.

Geometrically, one can see whether it is a max or a min, and why the gradient condition is telling.

How to judge analytically?

So what if we have multiple constraints?

ⓐ Each constraint tends to reduce the number of independent variables by 1.

ⓑ Each constraint tends to reduce the dimension of the space we evaluate the objective fnc on by 1.

ⓒ In $\mathbb{R}^3$, one objective fnc has level sets which are surfaces.

Each constraint does also; 2 constraints (2 surfaces) typically intersect in a line.

We then look for extreme of objective fnc along a line.

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ex. Find extrema of $f(x,y,z) = 2x + y^2 - z^2$

Subject to $g_1(x,y,z) = x - 2y = 0$

$g_2(x,y,z) = x + z = 0$

<u>Thm 4.3.2</u> Let $X \subset \mathbb{R}^n$ be open set

$f, g_1, \dots, g_k : X \to \mathbb{R}$ be C^1-functions. ($k<n$)

Let $S = \{\bar{x} \in X \mid g_1(\bar{x}) = c_1, \dots, g_k(\bar{x}) = c_k\}$

If $f|_S$ has an extremum at $\bar{x}_0 \in S$, where

$\nabla g_1(\bar{x}_0), \dots, \nabla g_k(\bar{x}_0)$ are <u>linearly indep.</u>

<u>as vectors</u>, then there exist scalars

$\lambda_1, \dots, \lambda_k$ ∋

$$\nabla f(\bar{x}_0) = \lambda_1 \nabla g_1(\bar{x}_0) + \dots + \lambda_k \nabla g_k(\bar{x}_0)$$

<u>Notes</u>: ① Recall lin. indep also means all nonzero!

Analytically $\nabla f(\vec{x}) = \begin{bmatrix} 2 \\ 2y \\ -2z \end{bmatrix}$ $\nabla g_1(\vec{x}) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ $\nabla g_2(\vec{x}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

~~Systems~~ ~~the~~ the constraint vectors are lin. indep. everywhere, so system is:

$$\nabla f(\vec{x}) = \lambda_1 \nabla g_1(\vec{x}) + \lambda_2 \nabla g_2(\vec{x})$$

$$\begin{bmatrix} 2 \\ 2y \\ -2z \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x - 2y = 0$$

$$x + z = 0$$

$$\left.\begin{aligned} 2 &= \lambda_1 + \lambda_2 \\ 2y &= -2\lambda_1 \\ -2z &= \lambda_2 \\ x &= 2y \\ x &= -z \end{aligned}\right\} \Rightarrow$$

① $2 = -y - 2z$

$\left.\begin{aligned} x &= 2y \\ x &= z \end{aligned}\right\}$ $2y - z = 0$

② $(2 = -y - 2z) \cdot 2$

$(0 = 2y - z)$

③ $\left.\begin{aligned} 4 &= -2y - 4z \\ 0 &= 2y - z \end{aligned}\right\}$ $\dfrac{4 = -5z}{\boxed{z = -\frac{4}{5}}}$ ...

See Mathematica