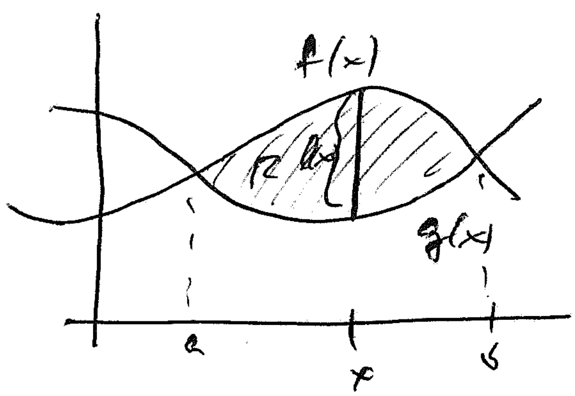
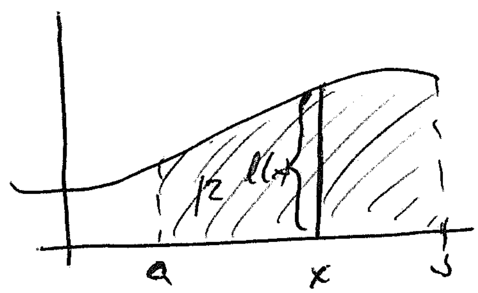
Sections 5.1 and 5.2 and 5.3

The area of a 2-dim region (say the difference between 2 fncs of a single variable x;



is just the "sum" of the lengths of all of the lines formed by slicing up the region perpendicular to the variable x.

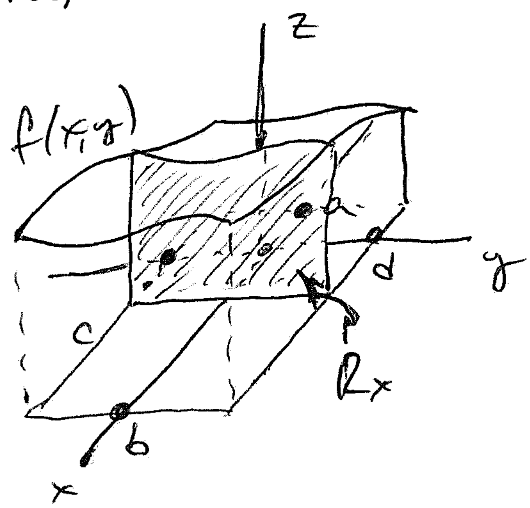


$$\text{Area}(R) = \int_a^b l(x)\,dx$$

$$\text{top} = \int_a^b (f(x) - g(x))\,dx$$

$$\text{bottom} = \int_a^b (f(x) - 0)\,dx$$

This is also true in higher dimensions.



$$S = \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} a \le x \le b \\ c \le y \le d \\ 0 \le z \le f(x,y) \end{array} \right\}$$

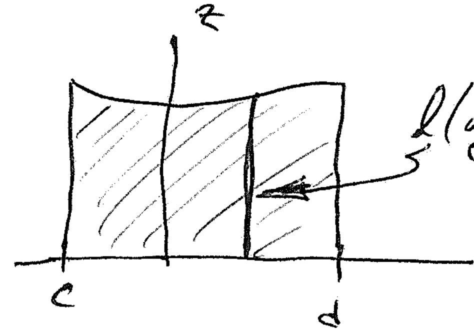
Vol(S) = sum of all areas of slices parameterized by x

$$\text{Vol}(S) = \int_a^b \text{Area}(R_x)\,dx$$

(Note: This may also equal $\text{Vol}(S) = \int_c^d \text{Area}(R_y)\,dy$)

So what is Area (R_x)? At a pt $x = x_0$,

$$\text{Area}(R_{x_0}) = \int_c^d \ell(y)\,dy = \int_c^d f(x_0, y)\,dy$$



Together, we get the "double integral"

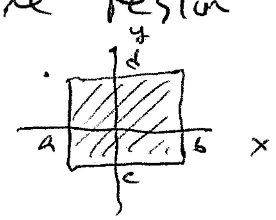
$$\text{Vol}(S) = \int_a^b \text{Area}(R_x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

<u>Notes</u> ① Parentheses are not needed; it is understood integrals are "nested".

② Subscript in integrand's x-variable also not needed, since x is "held fixed" on inside integral (like for partial differentiation).

③ If the limits of the y-variable (the inside integral here) do not depend on x, then the region ~~we~~ integrate over is rectangular.

Here region is



and $\text{Vol}(S) = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$

The notion of parameterizing the parallel slices through a solid to find volume is Cavalieri's Principle

Works in n-dim also!

Let S be an n-dim solid in $\mathbb{R}^n$ bounded in the x_1-dir by $[a,b]$. Then

$$\mathrm{Vol}(S) = \int_a^b \mathrm{Vol}(R_x)\,dx$$

where $\mathrm{Vol}(R_x)$ is the vol of the $(n-1)$-dim slice through S at $x \in [a,b]$.

Recursively, this will give us a nested n-tuple integral.

One can *define* the double integral on a *rectangular* region R via a 2-d Riemann Sum:

Define $f(x,y)$ on $R = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} a \le x \le b \\ c \le y \le d \end{array} \right\}$

and partition R into boxes:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$
$$c = y_0 < y_1 < \cdots < y_{m-1} < y_m = d.$$

So that $\Delta x_i = [x_i - x_{i-1}]$, $\Delta y_j = [y_j - y_{j-1}]$

and the area of the ijth box is $\Delta A_{ij} = \Delta x_i \Delta y_j$

Choose $\vec{p}_{ij} = (p_i, q_j)$, where $p_i \in [x_{i-1}, x_i]$
$q_j \in [y_{j-1}, y_j]$

for each $i = 1, \ldots, n$, $j = 1, \ldots, m$

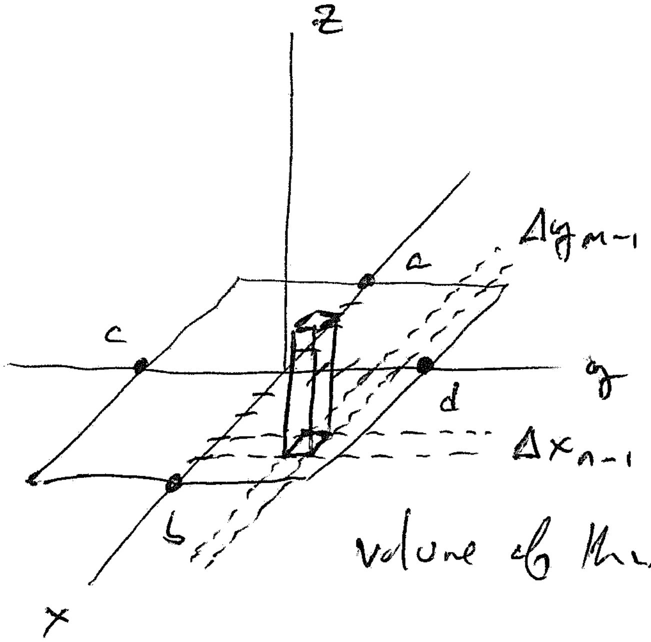
Then we can approximate the volume of the solid S formed by floor R and height graph$(f(x,y))$ as

$$Vol(S) \simeq \sum_{i=1}^{m} \sum_{j=1}^{m} f(p_{ij}) \Delta A_{ij}$$ ← length × width × height

This is a 2-d Riemann Sum.

Def The double integral of f on R (if it exists) is

$$\iint_R f\, dA = \cancel{\lim_{n,m \to \infty} \sum f(p_{ij}) \Delta A_{ij}}$$
$$= \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_{ij}) \Delta A_{ij}$$



(length · width · height)
volume of this box is $\Delta x_{n-1} \cdot \Delta y_{m-1} \cdot f(p_{ij})$

where $p_{ij} = (p_i, p_j)$, $p_i \in [x_{n-2}, x_{n-1}]$
$p_j \in [y_{m-2}, y_{m-1}]$.

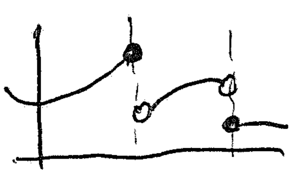
<u>Notes</u> ① If this limit exists then we say f is <u>integrable</u> on R.

② Also write

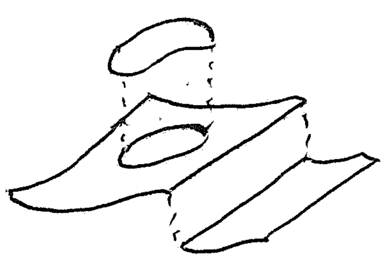
$$\iint_R f(x,y)\,dx\,dy \text{, or } \underbrace{\int_c^d \underbrace{\int_a^b f(x,y)\,dx}_{x}\,dy}_{y}.$$

③ If $f(x,y) < 0$ we ~~interpret~~ contribution to volume as negative! (like Calc I)

④ Also like Calc I, same problems and caveats exist with the limit existing.



- In Calc I, piecewise cont fncs on $[a,b]$ are integrable (w/ a finite # of discontinuities)



- Here, if f is bounded on R with the set of all discontinuities having 0-area, then f is integrable (think of graph(f) being smooth but cut up into a finite # of pieces).

But, if f is cont. on $\mathbb{R}$, then f is integrable on R (no discontinuities)

Thm (Fubini) Let f be bounded on $R = [a,b] \times [c,d]$ and assume the set S of discontinuities of f on R has 0-area. If every line parallel to the coordinate axes meets S in at most a finite number of places, then

$$\iint_R f\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Notes ① The fact that R is a rectangle is important!

② The fact that parallel lines (slices) meet S in at most a finite # of places is sufficient but not necessary. It forces the slice function to be piecewise cont. and thus integrable. But this is not the only way to have an integrable fnc on each slice.

③ for all intents and purposes, 0-area means smaller dimensional sets....

Properties of double integrals reflect those of their 1-d cousins

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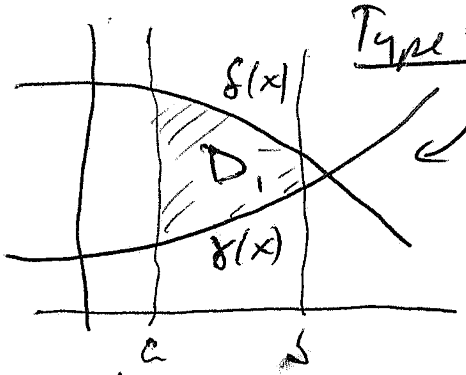
<u>Note</u>: Cavalieri's Principle holds for more general regions but the order of integration may matter!

<u>Def</u> A region $D \subset \mathbb{R}^2$ is called <u>elementary</u> if it can be described via an interval in one variable and as the difference between 2 fncs of that variable in the other.

<u>Type 1</u>



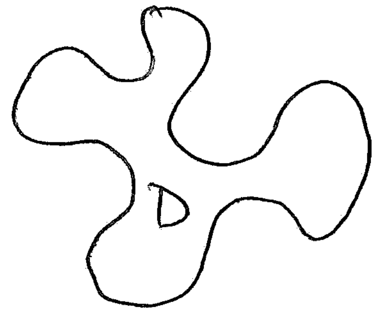
$D_1 = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, \gamma(x) \le y \le \delta(x)\}$

Type 3 if both

<u>Type 2</u>

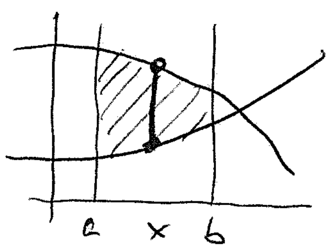
$D_2 = \{(x,y) \in \mathbb{R}^2 \mid c \le y \le d, \alpha(y) \le x \le \beta(y)\}$

Neither if neither



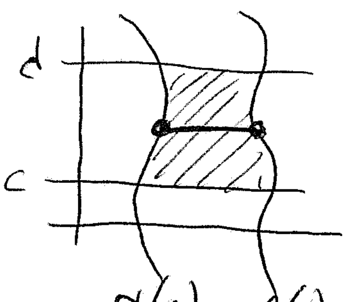
<u>Thm 5.2.10</u> If D is elementary in $\mathbb{R}^2$ and f cont. on D, then

① If Type 1, then



$$\iint_D f\, dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y)\, dy\, dx$$

② If type 2, then

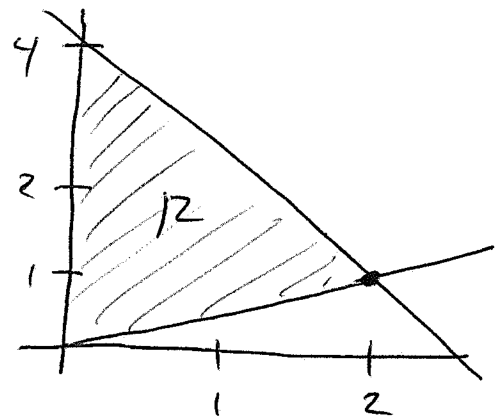


$$\iint_D f\, dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)\, dx\, dy$$

<u>pf</u> uneventful. Build out f on a rectangular region by declaring it 0 (discontinuous) outside of D and then use Fubini's thm.

ex Let $f(x) = \frac{-3}{2}x + 4$, $g(x) = \frac{1}{2}x$.

Declare R to be the region between from $x=0$ to $x=2$:

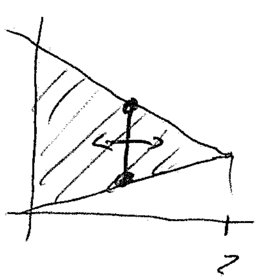


Integrate $f(x,y) = 2x+2y$ on R.

Solution: Show R is elementary and use previous thm to set up integral. Then use FTC from Calc I to do the integral calculation.

Solution ① R is Type I elementary on $[0,2]$ with

$$\delta(x) = \frac{-3}{2}x+4, \quad \gamma(x) = \frac{1}{2}x.$$



$$\iint_R f\,dA = \int_0^2 \int_{\frac{1}{2}x}^{\frac{-3}{2}x+4} (2x+2y)\,dy\,dx$$

$$= \int_0^2 \left[2xy + y^2 \Big|_{\frac{1}{2}x}^{\frac{-3}{2}x+4} \right] dx$$

$$= \int_0^2 \left[2x\left(\frac{-3}{2}x+4\right) + \left(\frac{-3}{2}x+4\right)^2 - \left(2x\left(\frac{1}{2}x\right) + \left(\frac{1}{2}x\right)^2\right)\right] dx$$

$$= \int_0^2 \left[-3x^2 + 8x + \frac{9}{4}x^2 - 12x + 16 - x^2 - \frac{1}{4}x^2 \right] dx$$

$$\iint_R f\,dA = \int_0^2 (-2x^2 - 4x + 16)\,dx \qquad \text{(Calc I int now)}$$

$$= \left[-\frac{2x^3}{3} - 2x^2 + 16x \Big|_0^2 \right] = -\frac{16}{3} - 8 + 32 = \frac{56}{3}.$$

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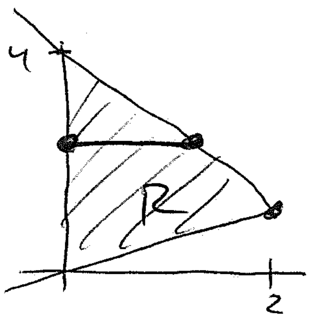
② R is Type II elementary on $[0,4]$ with

$$\alpha(x) = 0, \quad \beta(y) = \begin{cases} -\frac{2}{3}(y-4) & y \in [1,4] \\ 2y & y \in [0,1]. \end{cases}$$



$$\iint_R f\,dA = \int_0^4 \int_0^{\beta(y)} (2x + 2y)\,dx\,dy$$

$$= \int_0^1 \int_0^{2y} (2x+2y)\,dx\,dy + \int_1^4 \int_0^{-\frac{2}{3}(y-4)} (2x+2y)\,dx\,dy$$

$$= \int_0^1 \left[x^2 + 2xy \Big|_0^{2y} \right] dy + \int_1^4 \left[x^2 + 2xy \Big|_0^{-\frac{2}{3}(y-4)} \right] dy.$$

$$= \int_0^1 \left((2y)^2 + 2(2y)y\right) dy + \int_1^4 \left(\left(-\tfrac{2}{3}(y-4)\right)^2 + 2\left(-\tfrac{2}{3}(y-4)\right)y \right) dy.$$

$$= \int_0^1 8y^2\,dy - \frac{2}{3} \int_1^4 \left(-\tfrac{2}{3}(y^2 - 8y + 16) + 2y^2 - 8y \right) dy.$$

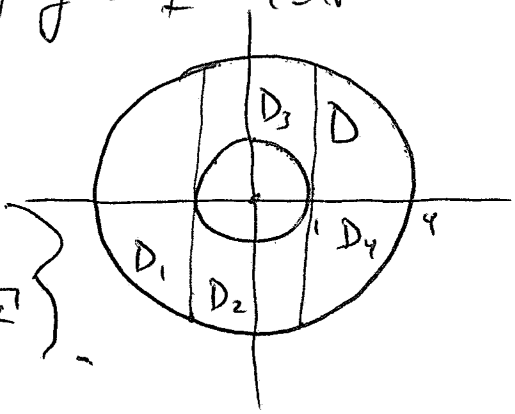
$$= \left[\frac{8}{3} y^3 \Big|_0^1 \right] - \frac{2}{3} \int_1^4 \left(\frac{4}{3} y^2 - \frac{8}{3} y - \frac{32}{3} \right) dy$$

$$= \frac{8}{3} - \frac{8}{9} \left[\frac{y^3}{3} - y^2 - 8y \Big|_1^4 \right] = \frac{8}{3} - \frac{8}{9} \left[\frac{64}{3} - 16 - 32 - \frac{1}{3} + 1 + 8 \right]$$

$$= \frac{8}{3} - \frac{8}{9}(-18) = \frac{8}{3} + 16 = \frac{56}{3} \quad \checkmark.$$

<u>Lastly</u>, more complicated regions can usually be broken up into a sum of elementary regions

ex The region D between $x^2+y^2=1$ and $x^2+y^2=4$



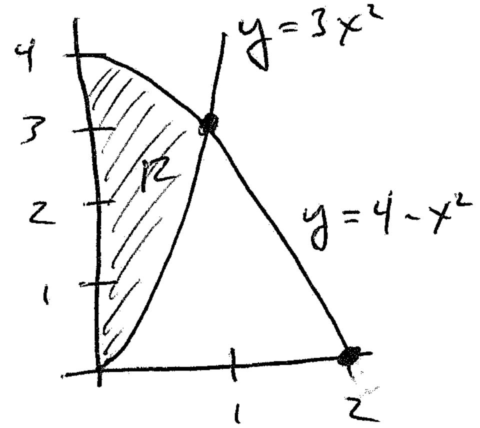
$$D_1 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} -4 \le x \le -1 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{array} \right\}.$$

$$D_2 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} -1 \le x \le 1 \\ -\sqrt{4-x^2} \le y \le -\sqrt{1-x^2} \end{array} \right.$$

$$D_3 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} -1 \le x \le 1 \\ \sqrt{1-x^2} \le y \le \sqrt{4-x^2} \end{array} \right.$$

$$D_4 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 1 \le x \le 4 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{array} \right.$$

ex. Integrate $f(x,y) = x^2 y$ over R.



Here, R is of type 3, and

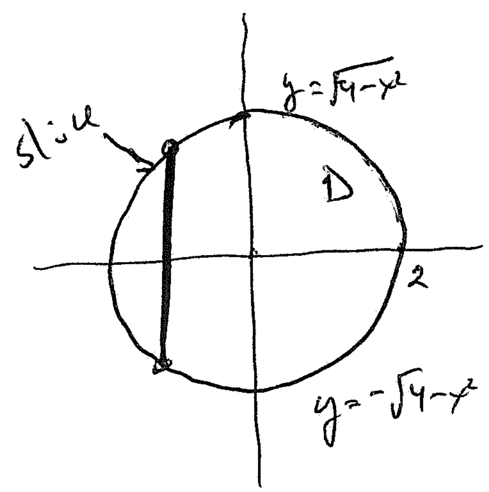
$$R = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} 0 \le x \le 1 \\ 3x^2 \le y \le 4-x^2 \end{matrix} \right\}$$

and

$$R = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} 0 \le x \le \sqrt{\frac{y}{3}} \\ 1 \le y \le 3 \end{matrix} \right\} \cup \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} 0 \le x \le \sqrt{4-y} \\ 3 \le y \le 4 \end{matrix} \right\}.$$

... Now do the integration ...

ex. Find the area of a circle of radius 2.



Here we center it at the origin as

$$x^2+y^2=4.$$

D is type 3, and

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} -2 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{matrix} \right\}.$$

Then

$$\text{Area}(D) = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, dx = 2\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

Q: Why is the integrand 1?