Change of variables (Section 5.5).

Q: Why not simply integrate the constant function $\rho = \text{const}$ in spherical coords to get $\mathrm{vol}(S^2(\rho))$?

All that would be needed is a change of coords on $\mathbb{R}^3$ for this....

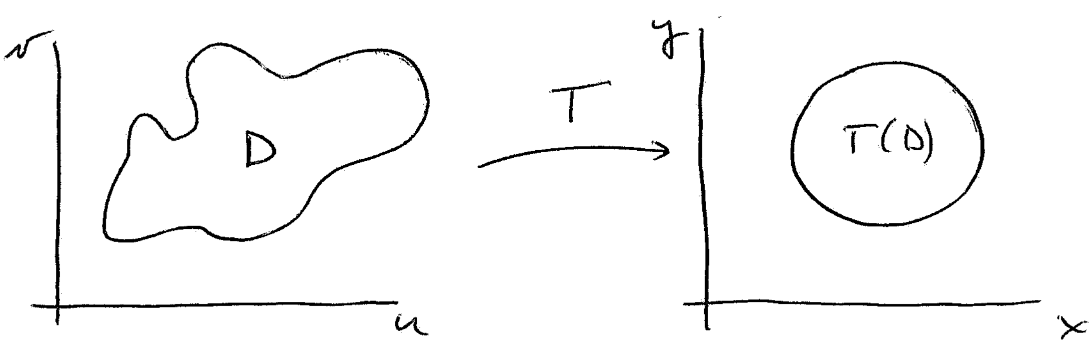
$$x = r\cos\theta\sin\varphi = x(r,\theta)$$
$$y = r\sin\theta\sin\varphi = y(r,\theta)$$
$$z = r\cos\varphi = z(r,\theta)$$

Placing new coords on a space involves again a function....

In $\mathbb{R}^2$, let $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(u,v) = (x(u,v), y(u,v))$ be a C^1-map (cont. diff).

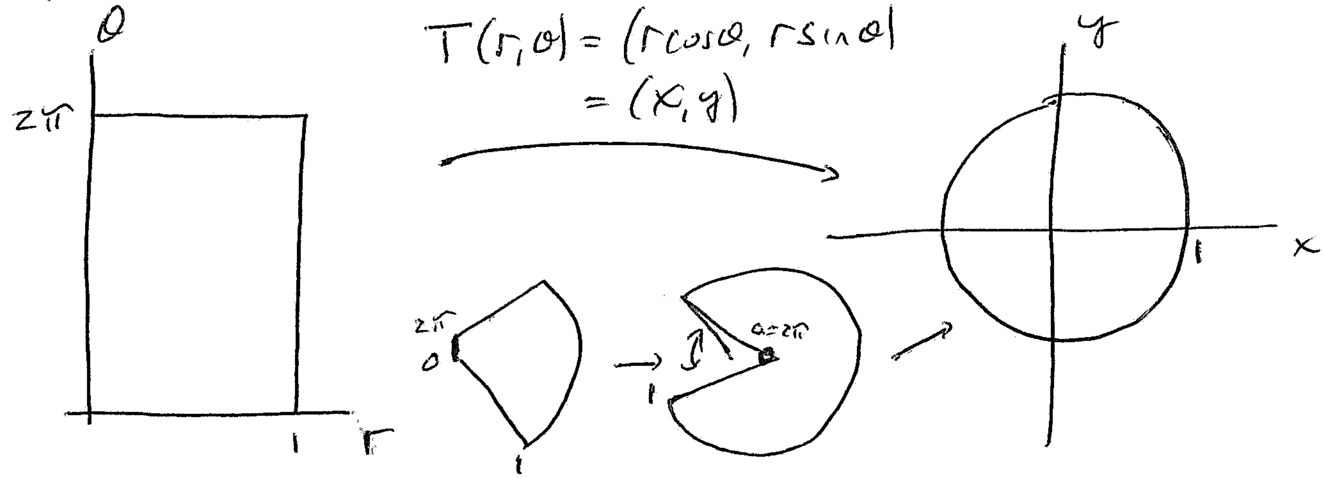
Then any subset $D \subset \mathbb{R}^2$ (in uv-coords) is mapped to another region $T(D)$ (in xy-coords).

abstract



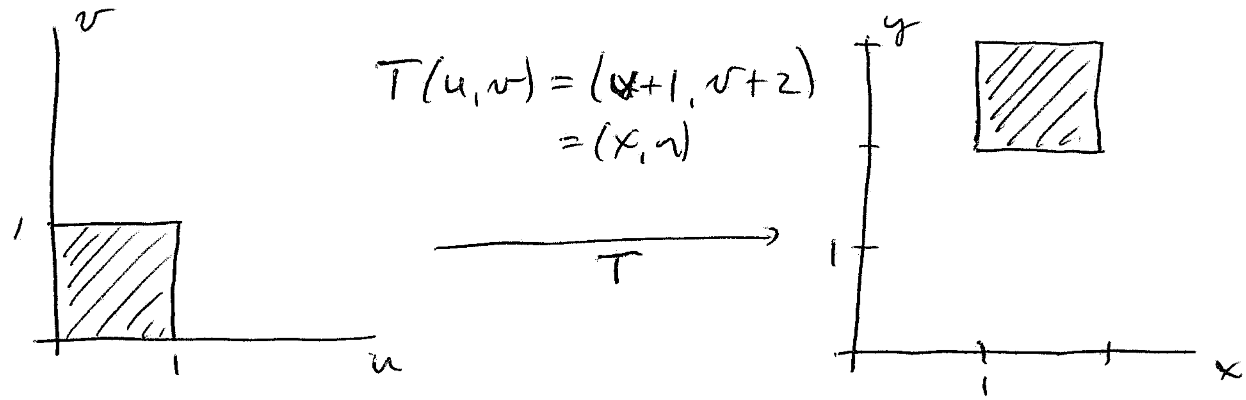
ex.

$$T(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$$



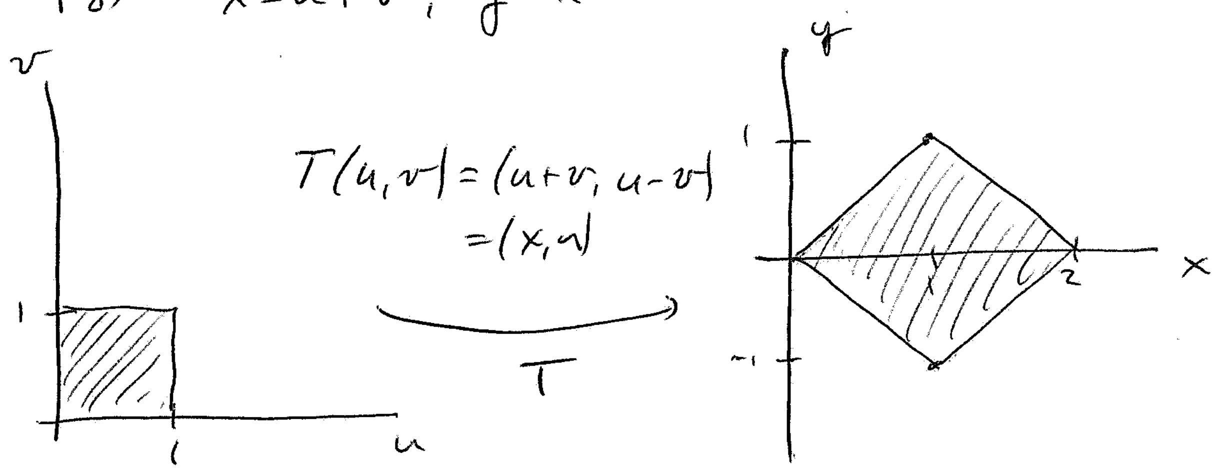
ex. Let $u = x+1$, $y = v+2$ be a translation. Then

$$T(u,v) = (u+1, v+2) = (x,y)$$



ex For $x = u+v$, $y = u-v$

$$T(u,v) = (u+v, u-v) = (x,y)$$



Here, regions on the left are "nicer" than those on the right.

Notes ① In general, it takes practice to "see" a transformation.

② Perhaps try to study the boundary first.

③ One type of transformation is easy to see:

Prop Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\det A = ad - bc \neq 0$.
Then $T(u,v) = A\begin{bmatrix} u \\ v \end{bmatrix}$ is 1-1, onto, takes parallelograms to parallelograms (vertices to vertices), and

$$\text{Area}(T(D)) = |\det A| \, \text{Area}(D).$$

Notes ① This is called a linear transformation. So $(0,0) \longmapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (No translation)

② Of course, works in n-dimensions.

Q: So why transform a region via a variable change?

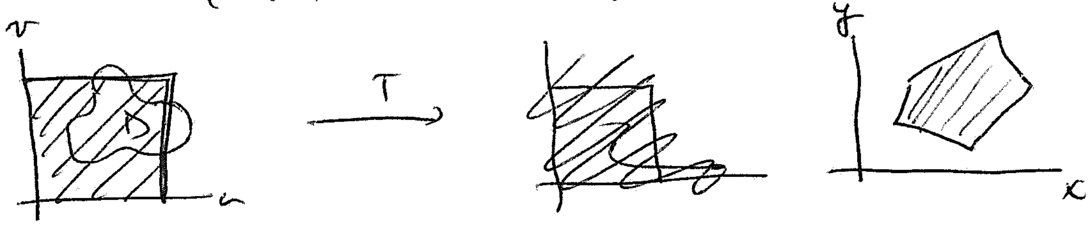
A: To make integration more straightforward (integrating over a rectangle is easier that over an elementary region that is more complicated.).

Special Note

Here, the more complicated region is in the codomain of the transformation $T(u,v)=(x,y)$.

This is because one writes the given variables x and y in ~~for~~ functional terms of new variables u and v, say. Hence $x=x(u,v)$ and $y=y(u,v)$ looks like

$$T(u,v)=(x(u,v),\, y(u,v))$$



You have seen this before in the 1-d Calculus I as the "Substitution Method"

If $g'(x)$ is cont. on $[a,b]$, and f is cont on the range of $u=g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Here, one changed variables to simplify the integrand and the substitution required the differential $du = g'(x)\, dx$.

We will need this also in more than 1-dim:

Let $T(u,v) = (x(u,v), y(u,v))$ be a C^1 transf. of $\mathbb{R}^2$.

Then $DT(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$, and

$$\mathrm{Jac}(T) = \det(DT) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \quad \text{as } \underline{\text{functions}}.$$

<u>Thm</u> Let D and D^* be elementary regions in the xy-plane and uv-plane (resp.) and suppose $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is C^1, $D = T(D^*)$, and T is 1-1 on (the interior of) D^*.

$\Longrightarrow$ for any integrable $f: D \longrightarrow \mathbb{R}$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

<u>Fun Fact</u>: By this theorem, you learned the Substitution Method backwards.:

In dim 1:

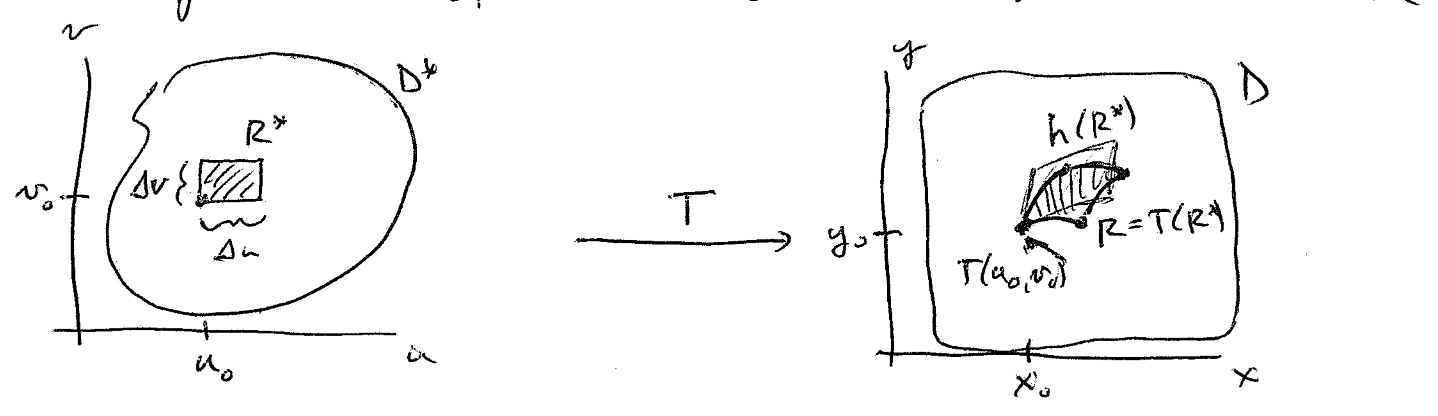
$$\int_I f(x)\,dx = \int_{I^*} f(x(u))\,x'(u)\,du \,, \quad \text{where } I=[a,b], \; I^*=[u(a),u(b)]$$

and the 1-d Jacobian was $x'(u)$ for the coordinate change $T(u)=x(u)$.

- You learned it backwards because your purpose was to simplify the integrand
- The purpose in multidimensional calculus is to simplify the integrating region.

Q: Why does the Jacobian arise in the equation here?

A: Under the transformation T, a small area given by $\Delta u = u-u_0$, $\Delta v = v-v_0$ in D^* goes to a new one



Since T is C^1, $R = T(R^*)$ is a small region with area area(R). A linear approximation to $R = T(R^*)$ is $h(R^*)$ given by

$$h(u,v) = T(u_0,v_0) + DT(u_0,v_0)\begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix}$$

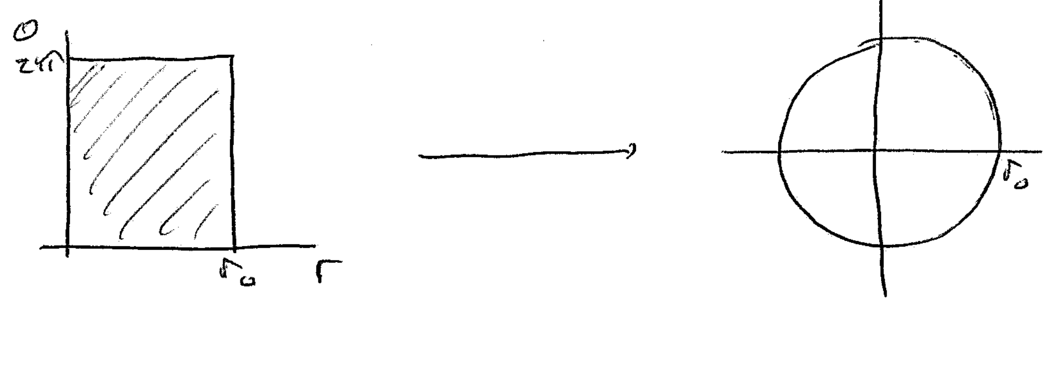
This linear approx takes $(u_0,v_0) \longmapsto T(u_0,v_0) = (x_0,y_0)$ and takes R^* to a parallelogram close to $R = T(R^*)$ where

$$\text{area}(R) \approx \text{area}(h(R^*))$$

Here area$((R^*))$ is $\left\| \begin{bmatrix} \Delta u \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ \Delta v \end{bmatrix} \right\| = \Delta u \Delta v$

$$\begin{aligned} \text{area}(h(R^*)) &= \left\| h\left(\begin{bmatrix} \Delta u \\ 0 \end{bmatrix}\right) \times h\left(\begin{bmatrix} 0 \\ \Delta v \end{bmatrix}\right) \right\| \\ &= \left| \det DT(u_0,v_0) \right| \cdot \text{area}(R^*) \\ &= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v \end{aligned}$$

Now consider a change of coordinates from polar to cartesian: $x = r\cos\theta$, $y = r\sin\theta$



Here $dA = dx\,dy = \left|\dfrac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$

$$= \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix} dr\,d\theta = r\,dr\,d\theta$$

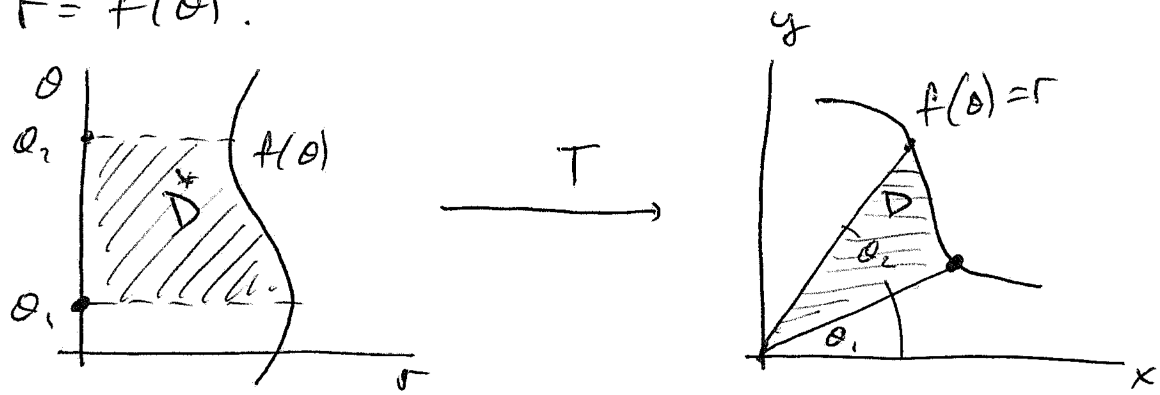
Hence $\displaystyle\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$

↑ This is where the r comes from

But here T is not 1–1 on D^* !! But still works.

Why?

Let $r = f(\theta)$.



Here, what is the area of D from Calculus II?

Here $\text{area}(D) = \iint_D 1 \, dx\,dy = \iint_{D^*} r\,dr\,d\theta$

$$= \int_{\theta_1}^{\theta_2} \int_0^{f(\theta)} r\,dr\,d\theta = \int_{\theta_1}^{\theta_2} \frac{r^2}{2}\Big|_0^{f(\theta)} d\theta$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{2}[f(\theta)]^2 \, d\theta \quad \text{(formula in Calc II)}.$$

It turns out that the Transformation T in Thm need not be 1-1 on the boundary of D^*. As long as the boundary of D^* is a "nice" curve, its contribution to the integral is ~~zero~~ 0, and can be neglected.

Nothing changes in higher dim except for the dim:

$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$$

taking D^* to D, with

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

as the extra piece in the integrand