Section 6.1

Notice the notation:

- In Calc I, $\int_a^b f(x)\,dx = \int_I f\,dx$, where $x \in I \subset \mathbb{R}$ and f is a function of x.
- In Calc III so far:
  - Double Integrals: $\iint_D f\,dA$, $D \subset \mathbb{R}^2$
  - Triple Integrals: $\iiint_W f\,dV$, $W \subset \mathbb{R}^3$

    and f is real-valued $f: W \to \mathbb{R}$.

These are definite integrals, and the notation denotes a coordinate-free way to write the quantities: For D in the xy-plane,

$$\iint_D f\,dA = \iint_D f(x,y)\,dx\,dy.$$

Here, we define new ways to study properties of functions over relevant domains in $\mathbb{R}^n$.

Line Integrals

(I) Real-valued (scalar) functions

Recall differentiating a C^1-func $f : X \subset \mathbb{R}^n \to \mathbb{R}$ along a curve $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$, (where $\vec{x}(I) \subset X \subset \mathbb{R}^n$) looks like Calc I:

$$\frac{df}{dt}(t) = \frac{d}{dt} f(\vec{x}(t)) = Df(\vec{x}(t)) \cdot \frac{d\vec{x}}{dt}(t)$$

$$= \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{dt} \in \mathbb{R}.$$

One would expect integrating a function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ over a curve (adding its values over the curve) should also seem 1-dim.

In fact, it is quite similar

<u>Def</u> Given an integrable $f : X \subset \mathbb{R}^n \to \mathbb{R}$, and a C^1 $\vec{x} : [a,b] \to \mathbb{R}^n$, where $\vec{x}([a,b]) \subset X$, the <u>scalar line integral</u> of f over $\vec{x}$ is

$$\int_{\vec{x}} f\, ds = \int_a^b f(\vec{x}(t)) \|\vec{x}\,'(t)\|\, dt$$

Notes ① The symbolic s denotes arc-length:

For any curve parameterization $\vec{x}:[a,b]\to\mathbb{R}^n$ the arc-length parameter is

$$s(t)=\int_0^t \|\vec{x}'(\tau)\|\,d\tau$$

Seen as a change of variables, the new differential is then (by FTC)

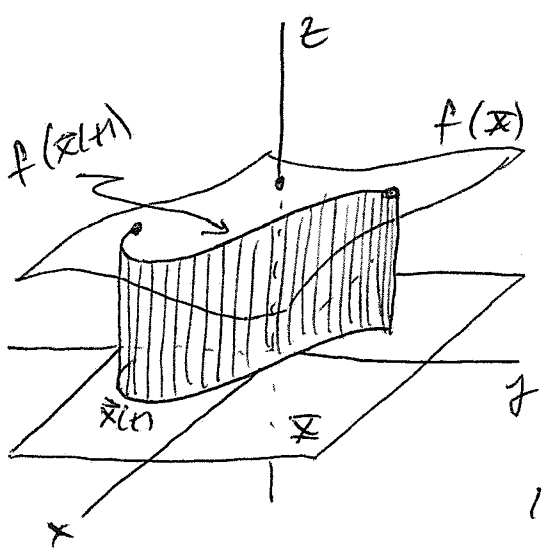
$$ds=\frac{d}{d\tau}\left(\int_0^t \|\vec{x}'(\tau)\|\,d\tau\right)dt = \|\vec{x}'(t)\|\,dt$$

This suggests that the (scalar) line integral is parameterization independent.

This suggestion is correct.

② Also sometimes called the path integral. or line integral of a scalar field.

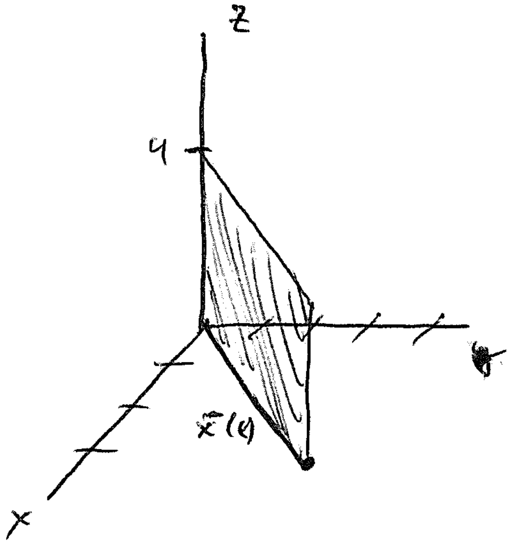
③ Geometric Interpretation: For $f\geq 0$ on $\vec{x}$,



$$\int_{\vec{x}} f\,ds = \text{area of the surface bounded by } f(\vec{x}([a,b])) \text{ and } \vec{x}([a,b]).$$

This is a 1-d calculation but in $\mathbb{R}^n$, $n>1$.

ex. Let $f(x,y) = 4$, $\vec{x}(t) = \begin{bmatrix}3\\4\end{bmatrix}t$, $t \in [0,1]$.



Here $\displaystyle\int_{\vec{x}} f\,ds = \int_0^1 f(\vec{x}(t))\,\|\vec{x}\,'(t)\|\,dt$

$$= \int_0^1 4\sqrt{9+16}\,dt$$

$$= 20t\Big|_0^1 = 20.$$

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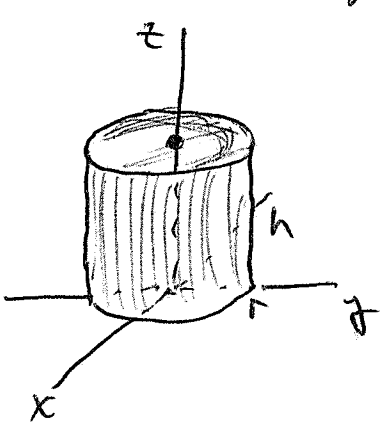
ex. The surface area of a cylinder of radius r and height h is $SA = 2\pi r h$. (no top or bottom).

We can functionally create this.

Let $g(x,y) = h$ over $\vec{x}(t) = \begin{bmatrix} r\cos t \\ r\sin t\end{bmatrix}$, $t \in [0, 2\pi]$



$$\int_{\vec{x}} g\,ds = \int_0^{2\pi} \underbrace{h}_{g(\vec{x}(t))} \underbrace{\sqrt{r^2\cos^2 t + r^2\sin^2 t}}_{\|\vec{x}\,'(t)\|}\,dt = \int_0^{2\pi} hr\,dt$$

$$= 2\pi r h.$$

Note: Like in Calc I, we are just adding up the values of f along the curve.

ex. Integrate $f(x,y,z) = xyz$ over $\vec{x} = \begin{bmatrix} t \\ 2t \\ 3t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} t$

for $t \in [0,2]$.

Solution: Here $\vec{x}\,'(t) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, so $\|\vec{x}\,'(t)\| = \sqrt{1+4+9} = \sqrt{14}$.

Then $\displaystyle\int_{\vec{x}} f\, ds = \int_0^2 t(2t)(3t)\sqrt{14}\, dt = \int_0^2 6\sqrt{14}\, t^3\, dt$

$$= \frac{3\sqrt{14}}{2} t^4 \Big|_0^2 = 24\sqrt{14}.$$ ∎

~~Def~~. II Vector-valued functions (vector fields)

For a curve $\vec{x} : [a,b] \to \mathbb{R}^n$ inside a vector field $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ ($\vec{x}([a,b]) \subset X$), one can ask how much of the vector field can be "seen" by (or act on) the curve $\vec{x}(t)$

Recall for a curve where $\vec{x}\,'(t) \neq 0 \;\; \forall t \in [a,b]$,

$\vec{T}(t) = \dfrac{\vec{x}\,'(t)}{\|\vec{x}\,'(t)\|}$ is the unit tangent vector at t.

Then the component of $\vec{F}$ along $\vec{x}$ is $\vec{F}(\vec{x}(t)) \cdot \vec{T}(t)$

We can add up these values along $\vec{x}$ as a function to get

$$\int_{\vec{x}} (\vec{F} \cdot \vec{T})\, ds,$$

the scalar line integral of $\vec{F} \cdot \vec{T}$ along $\vec{x}$.

(In a way this represents the aggregate boost or hindrance a particle feels by $\vec{F}$ along $\vec{x}$).

But

$$\int_{\vec{x}} (\vec{F} \cdot \vec{T})\, ds = \int_a^b \vec{F}(\vec{x}(t)) \cdot \frac{\vec{x}\,'(t)}{\|\vec{x}\,'(t)\|} \|\vec{x}\,'(t)\|\, dt$$

$$= \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt$$

$$= \int_a^b \vec{F}(\vec{x}(t)) \cdot d\vec{s} = \int_{\vec{x}} \vec{F} \cdot d\vec{s}$$

where

$$d\vec{s} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} = \begin{bmatrix} x_1'(t)\,dt \\ \vdots \\ x_n'(t)\,dt \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} dt = \vec{x}\,'(t)\, dt$$

is a vector infinitessimal displacement and represents an infinitessimal change in displacement along each coordinate direction (instead of just along the curve).

<u>Def.</u> For a C^1 curve $\vec{x}:[a,b]\to\mathbb{R}^n$ in a vector field $\vec{F}:X\subset\mathbb{R}^n\to\mathbb{R}^n$ (here $\vec{x}([a,b])\subset X$), the <u>vector-line integral</u> of $\vec{F}$ along $\vec{x}$ is

$$\int_{\vec{x}}\vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t))\cdot\vec{x}\,'(t)\,dt$$

<u>Notes</u> ① Recall that the work done by a vector field $\vec{F}$ on a particle is just force times displacement. As vector such thes (when the particle movement

- is linear: $W = \vec{F}\cdot\vec{d} = \|\vec{F}\|\,\|\vec{d}\|\cos\theta$

  $= \|\vec{F}\|\cdot$(displacement in direction of $\vec{F}$).

- is curved: Measured infinitessimally, and

  $$W = \int_a^b \vec{F}(\vec{x}(t))\cdot\vec{x}\,'(t)\,dt$$

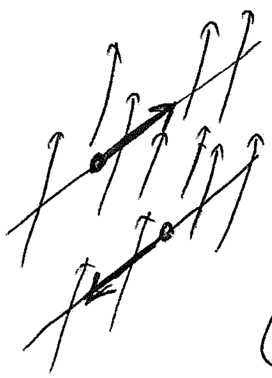
  is a scalar integral.

Notes (cont'd)

② Some facts:

<u>Thm</u> A scalar line integral is independent of its parameterization.

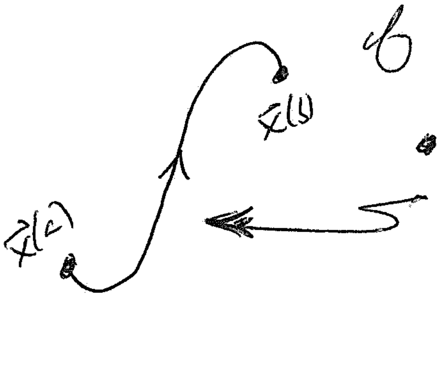
<u>Thm</u> A vector line integral depends on the parameterization <u>only</u> in the direction of travel.



③ Some curve facts:

ⓐ For $\vec{x}:[a,b]=I \to \mathbb{R}^n$ a piecewise C^1-curve, A C^1-function $h:[c,d]=J \to I$ which is 1-1, onto and has a C^1-inverse induces $\vec{p}: J \to \mathbb{R}^n$, $\vec{p} = \vec{x}\circ h$ a <u>reparameterization</u>.

ⓑ For $\vec{x}$ 1-1 on I, $\vec{x}(I)$ has only 2 directions of travel (increasing values of t).



- A choice of direction is called an <u>orientation</u> of the curve.

Notes ③ ⑤ cont'd.

- A reparameterization is called orientation preserving if the directions are the same. Else called orientation reversing.
- A curve is simple if $\vec{x} : [a,b] \to \mathbb{R}$ is injective on (a,b), and closed if $\vec{x}(a) = \vec{x}(b)$.
- In general, scalar line (path) integrals are defined on curves, and vector line integrals are defined on oriented curves.

④ If $\vec{x}$ is a closed simple curve, then the notation for a line integral (vector) is

$$\oint_{\vec{x}} \vec{F} \cdot d\vec{s}$$

and is called the circulation of $\vec{F}$ along $\vec{x}$.

We will see what this means in time.

Notes cont'd.

(5) For a ~~line inte~~ vector line integral $\int_{\vec{x}} \vec{F} \bullet d\vec{s}$

$$\vec{F} \bullet d\vec{s} = F_1\,dx_1 + \cdots + F_n\,dx_n = \sum_{i=1}^{n} F_i\,dx_i$$

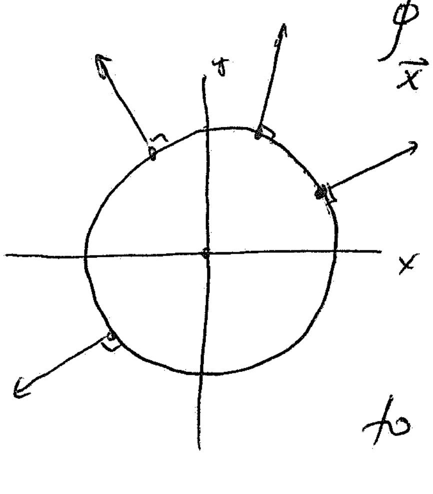
is called a differential 1-form.

ex. Calculate the vector line integral of

$$\vec{F} = x\vec{i} + y\vec{j} \quad \text{over} \quad \vec{x}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix},\ t \in [0, 2\pi]$$

Here, $\vec{x}(t)$ is the unit circle, a simple closed curve.

So



$$\oint_{\vec{x}} \vec{F} \bullet d\vec{s} = \int_0^{2\pi} \vec{F}(\vec{x}(t)) \bullet \vec{x}'(t)\,dt$$

$$= \int_0^{2\pi} \left(\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \bullet \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \right) dt = \int_0^{2\pi} 0\,dt = 0.$$

Here, the vector field is perpendicular to the curve everywhere. Hence $\int_{\vec{x}} (\vec{F} \bullet \vec{T})\,ds = 0$.

ex. And if the curve $\vec{x}(t)$ is an integral curve of $\vec{F}$?

By definition, $\vec{x}'(t) = F(\vec{x}(t))$ $\forall t \in [a, b]$, so

$$\int_{\vec{x}} \vec{F} \bullet d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \bullet \vec{x}'(t)\,dt = \int_a^b \vec{F}(\vec{x}(t)) \bullet F(\vec{x}(t))\,dt$$

$$= \int_a^b \| F(\vec{x}(t)) \|^2\,dt.$$

(6) In this case, we are simply adding up the effect of the vector field on the curve (the tangent component).

So also called a line integral of a vector field