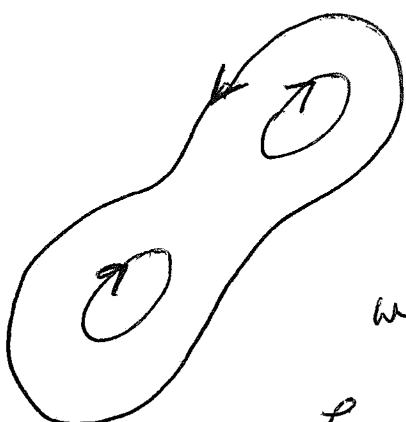
Section 6.2

<u>Green's Thm</u> Let D be a closed, bounded region in $\mathbb{R}^2$, whose boundary $\vec{c} = \partial D$ is a finite union of simple, closed curves, oriented so that D is always on the left.



For a C^1-vector field on D
$$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j},$$
we have

$$\int_{\vec{c}} \vec{F}\cdot d\vec{s} \overset{ⓐ}{=} \oint_{\vec{c}} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$
$$\overset{ⓑ}{=} \underbrace{\iint_D (\nabla\times\vec{F})\cdot\vec{k}\,dA}_{\text{you will see this again.}}.$$

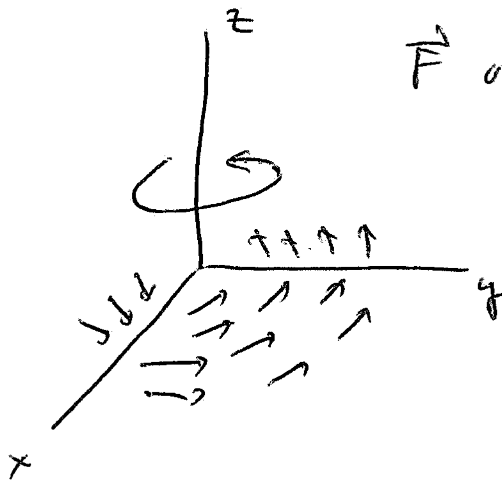
<u>Notes</u> ① ⓐ is obvious since $\vec{F} = \begin{bmatrix} M \\ N \end{bmatrix}$, $d\vec{s} = \begin{bmatrix} dx \\ dy \end{bmatrix}$.

ⓑ is also obvious since $(\nabla\times\vec{F}) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$

To see this, think of a vector field in $\mathbb{R}^2$ as a vector field in $\mathbb{R}^3$ with z-component 0. Then calculate $\nabla\times\vec{F}$.

Notes (cont'd)

② The theorem basically says that the vector line integral (the circulation) of $\vec{F}$ along ∂D equals the curl of $\vec{F}$ on D.

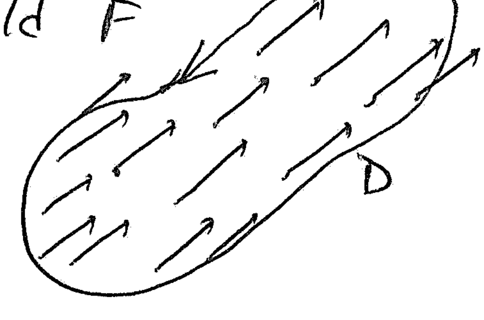


- $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$ measures the aggregate component of $\vec{F}$ tangent to $\vec{c}$
- curl in 2-d measures in $\mathbb{R}^2$ the rotation one would feel if flowing along $\vec{F}$.

So the sum total of the push or pull of a particle by $\vec{F}$ along ∂D equals to total rotation effect of $\vec{F}$ on D.

ex. For a constant vector field $\vec{F}$

- $\nabla \times \vec{F} = \vec{0}$, so RHS $= 0$.
- What happens on LHS?
- What happens in middle?



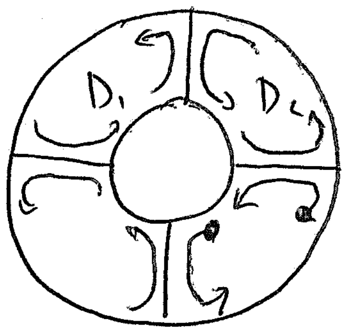
Notes (cont'd)

③ The proof is elementary, and relies on 3 facts:

Ⓐ Lemma 1 If D is elementary of type I then $\oint_{\partial D} M\,dx = -\iint_D \frac{\partial M}{\partial y}\,dA$

Ⓑ Lemma 2 If D is ~~of~~ elementary of type II then $\oint_{\partial D} N\,dy = \iint \frac{\partial N}{\partial x}\,dA$

Ⓒ Any region D, valid for Green's Thm, can be cut up into a finite # of elementary regions, so that



(i) the ends of each cut lie on ∂D

(ii) the cuts do not intersect

(iii) $D = \sqcup D_i$ with each D_i elem. of type III.

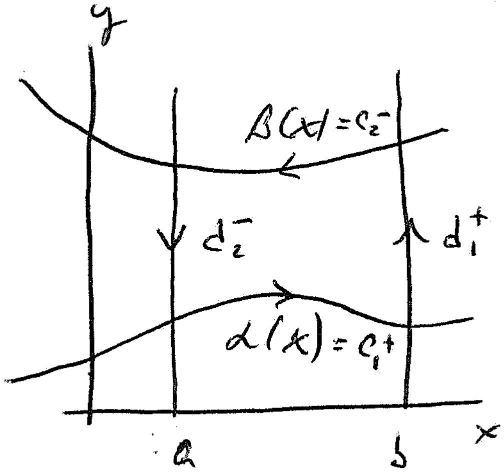
(iv) each cut intersects exactly 2 D_i's with each cut oriented in each D_i oppositely.

Note: Then vector line integrals along cuts will cancel out. So no contribution of cuts in the calculation. And for the double integral, the cuts provide no contribution either.

Idea of proof of Lemma 1

<u>Lemma</u> If D is elementary of type I,

then $\oint_{\partial D} M\,dx = -\iint_D \frac{\partial M}{\partial y}\,dA$



pf. Here $D = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} a \le x \le b \\ \alpha(x) \le y \le \beta(x) \end{array}\right.$

Orient $\partial D = c_1^+ \cup d_1^+ \cup c_2^- \cup d_2^-$ as needed. (plus means compatible with variable, minus contrary).

Then (RHS) $-\iint_D \frac{\partial M}{\partial y}\,dA = \int_a^b \int_{\alpha(x)}^{\beta(x)} -\frac{\partial M}{\partial y}\,dy\,dx$

$$= \int_a^b -(M(x,\beta(x)) - M(x,\alpha(x))\,dx$$

by FTC
$$= \int_a^b (M(x,\alpha(x)) - M(x,\beta(x)))\,dx$$

Here $-\int_a^b M(x,\beta(x))\,dx = \int_{c_2^-} M\,dx$, $\int_a^b M(x,\alpha(x))\,dx = \int_{c_1^+} M\,dx$

And $\int_{d_1^+} M\,dx = \int_{d_2^-} M\,dx = 0$ since x is constant here.

Thus $-\iint_D \frac{\partial M}{\partial y}\,dA = \int_{c_1^+} M\,dx + \int_{d_1^+} M\,dx + \int_{c_2^-} M\,dx + \int_{d_2^-} M\,dx$

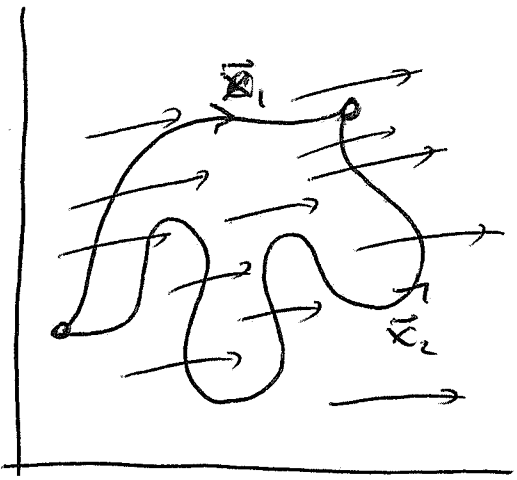
$$= \oint_{\partial D} M\,dx. \qquad \blacksquare$$

Exercise: Prove Lemma 2.

Section 6.3 Conservative Vector fields

We start with a definition

<u>Def.</u> A C^0-vector field $\vec{F}$ has path-independent line integrals if for any 2 piecewise C^1-curves w/ same endpts @ $\vec{x}_1$ and $\vec{x}_2$, we have $\int_{\vec{x}_1} \vec{F}\cdot d\vec{s} = \int_{\vec{x}_2} \vec{F}\cdot d\vec{s}$.



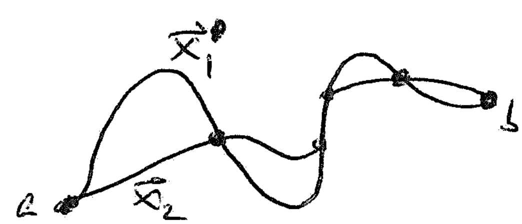
Notice that since $\vec{x}_1, \vec{x}_2$ have same endpts, they together form a closed curve (traversing $\vec{x}_2$ backwards, that is).

Sometimes this closed curve is simple.

Thm A C^0-vector field $\vec{F}$ has path-indep line integrals iff $\oint_{\vec{C}} \vec{F} \cdot d\vec{s} = 0 \quad \forall$ piecewise C^1 simple closed curves C in domain of $\vec{F}$.

Note: If $\vec{x}_1 \cup \vec{x}_2^-$ is not simple and consists of a finite # of isolated intersections or intervals that coincide.

Still true! (How?)



Def A C^0-vector field $\vec{F}$ is called conservative or a gradient field if $\exists$ C^1-real-valued function f $\ni$ $\vec{F} = \nabla f$. Such an f is called a potential for $\vec{F}$.

Notes ① Conservative vector fields always ~~have~~
(166) (and only) have path-indep. line
integrals

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_{\vec{x}} \nabla f \cdot d\vec{s} = \int_a^b \nabla f(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt$$

$$= \int_a^b \frac{d}{dt}\left[f(\vec{x}(t))\right] dt$$

$$\overset{FTC}{=} f(\vec{x}(t))\Big|_a^b = \underbrace{f(\vec{x}(b)) - f(\vec{x}(a))}_{\text{depends only on the end pts.}}$$

This is thm 6.3.3.

② In $\mathbb{R}^2$ or $\mathbb{R}^3$, conservative vector fields are irrotational: if $\vec{F}$ is conservative, then

$$\nabla \times \vec{F} = \nabla \times \nabla f = \vec{0}.$$

Converse: If $\vec{F}$ is irrotational, <u>and</u> <u>domain is simply connected</u>, then $\vec{F}$ is conservative.

<u>This is Thm 6.3.5</u>

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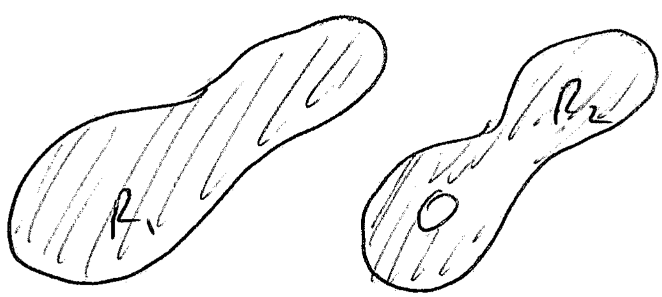
ex.

<u>Def</u> A ~~space~~ region is simply connected if it is connected (comes in one piece) and every ~~loop~~ simple closed curve in region has entire interior in region



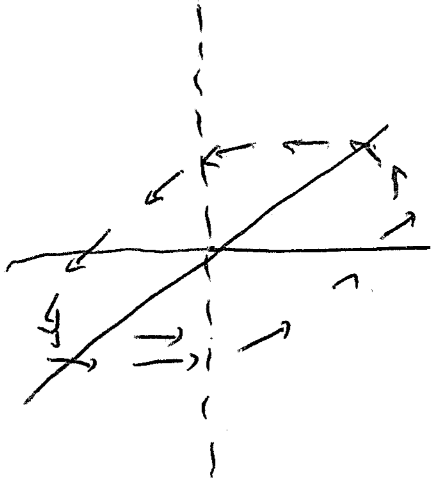
R_1 - simply connected
R_2 - not simply conn.

ex. Let $\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} + 0\vec{k}$ on

$W = \mathbb{R}^3 - \{(0,0,z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\}$. has

no curl ($\nabla \times \vec{F} = \vec{0}$) but $\vec{F}$ is

not conservative. Here W is not simply conn.



Q: How to tell if a $\vec{F}$ is conservative otherwise?

A: ① Check mixed partials of gradient

② Integrate to find f:

ex. (6.3.14) Find a potential for

$$\vec{F} = \begin{bmatrix} y+z \\ 2z \\ x+y \end{bmatrix}$$ if conservative.

Soln: If exists, then $\frac{\partial f}{\partial x} = y+z$, so $f(x,y,z) = xy + xz + g(y,z)$

$$\Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}(y,z) = 2z$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & 2z & x+y \end{vmatrix} = (1-2)\vec{i}$$

Not conservative $\nabla \times \vec{F} \neq \vec{0}$

ex. Find a potential for

$$\vec{F} = (2x+y)\vec{i} + (z\cos yz + x)\vec{j} + (y\cos yz)\vec{k}$$

Soln Here $\frac{\partial f}{\partial x} = 2x+y$, so

$$f(x,y,z) = x^2 + xy + g(y,z).$$

Hence $\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}(y,z) = x + z\cos yz$

$$\Rightarrow \frac{\partial g}{\partial y}(y,z) = z\cos yz$$

$$\Rightarrow g(y,z) = \sin yz + h(z)$$

$$= f(x,y,z) = x^2 + xy + \sin yz + h(z)$$

$$\frac{\partial f}{\partial z} = y\cos yz + h'(z) = y\cos yz$$

$$\Rightarrow h'(z) = 0 \Rightarrow h(z) = \text{const.}$$

$$\Rightarrow f(x,y,z) = x^2 + xy + \sin(yz)$$ ∎