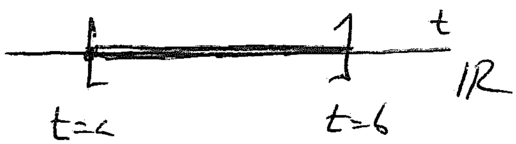
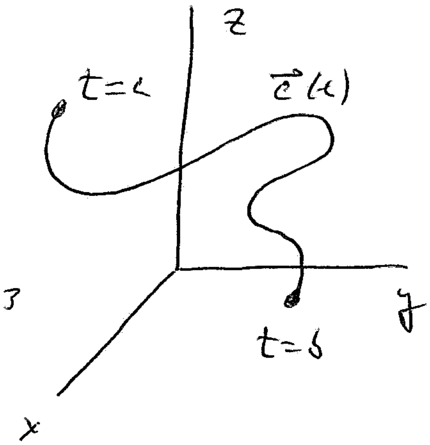
Section 7.1

We parameterize a curve in $\mathbb{R}^n$ via a map

$\mathbb{R}^3$ example



$\xrightarrow{\vec{c}}$

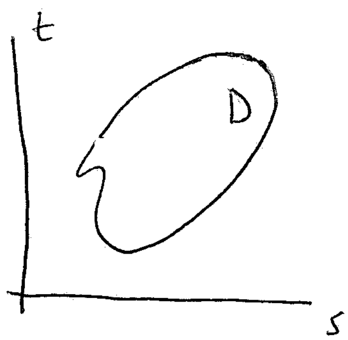


$$\vec{c}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \in \mathbb{R}^3$$

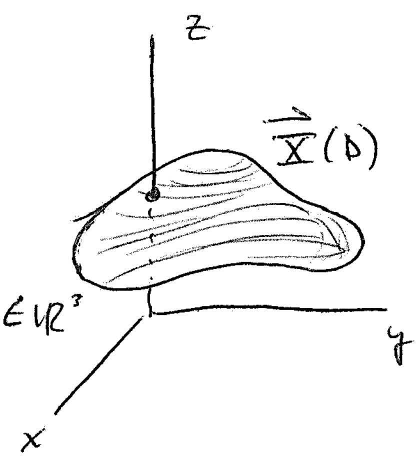
parametric eqns

Let $D \subset \mathbb{R}^2$ be a connected, open set along with some or all of its boundary pts. A parameterized surface in $\mathbb{R}^n$ is a C^0-function $\vec{X} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ that is 1-1 on $\mathring{D}$

$\mathbb{R}^3$ example



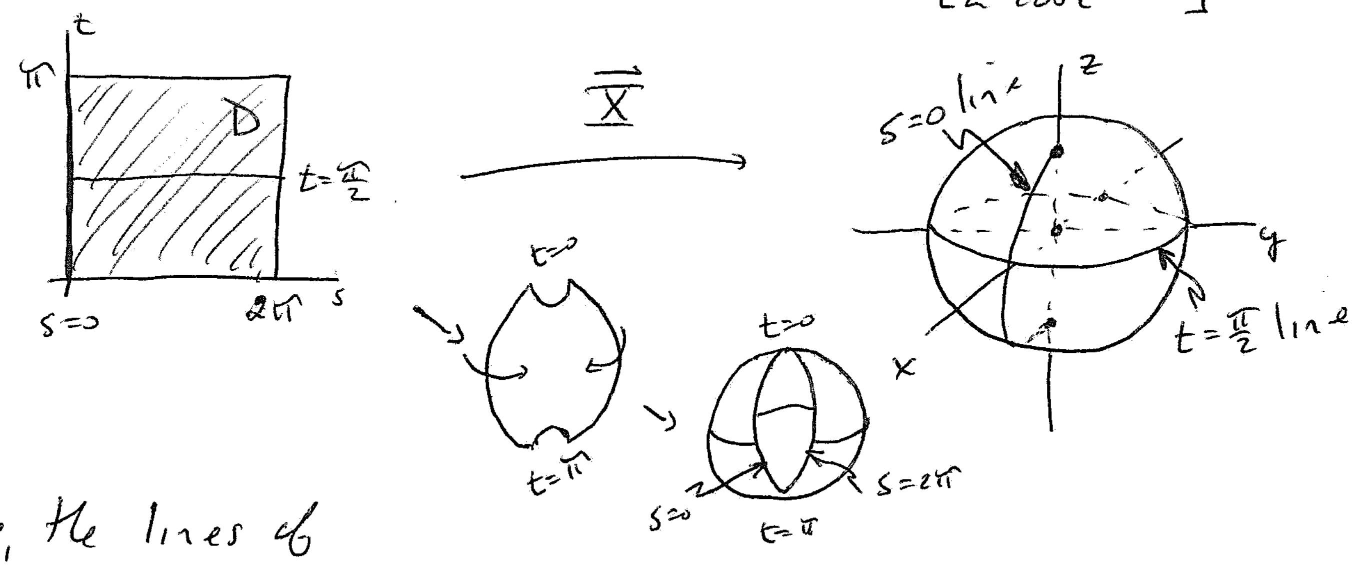
$\xrightarrow{\vec{X}}$



$$\vec{X}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} \in \mathbb{R}^3$$

parametric eqns

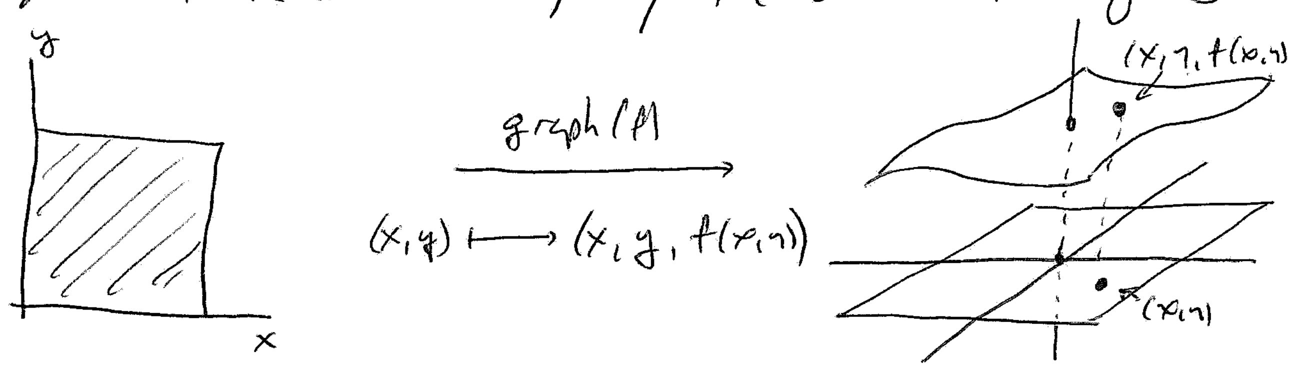
ex. Let $\vec{X}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\vec{X}(s,t) = \begin{bmatrix} a\cos s \sin t \\ a \sin s \sin t \\ a \cos t \end{bmatrix}$



Here, the lines of
are constant coordinate each
are the latitude and longitude lines of $S^2 \subset \mathbb{R}^3$.

Note: Just as a curve sits inside $\mathbb{R}^n$, $n>2$, a surface can have $\mathbb{R}^n$, $n>3$ as a codomain. As it is hard to see $\mathbb{R}^n$, $n>3$, the book and we will limit most of our examples to $\mathbb{R}^3$ surfaces.

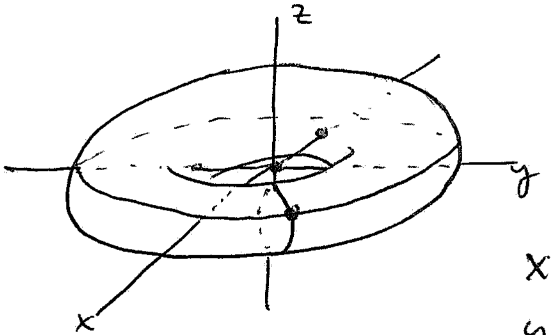
ex. Any $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ has a graph $\mathrm{graph}(f) \subset \mathbb{R}^3$, parameterized directly by the coordinates of D:



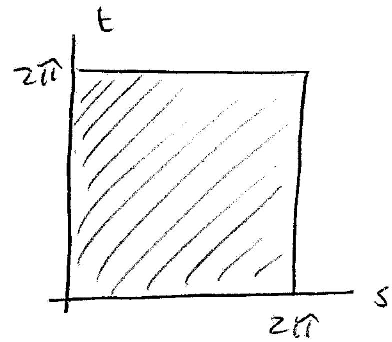
Example 5 is beautiful!

<u>Alternate example</u>: The Clifford Torus.

The 2-torus $\mathbb{T}^2$ has a nice description as $S^1 \times S^1$:

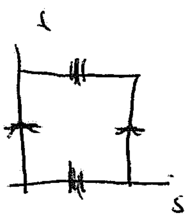


$\xleftarrow{\vec{X}}$



$x = (a + b\cos t)\cos s$
$y = (a + b\cos t)\sin s$
$z = b \sin t$

Here $a > b$, b is the cross sectional radius and a is the radial distance from the center of the torus to the z-axis.



The Clifford Torus is $\vec{X}(s,t) = \begin{bmatrix} a\cos s \\ a \sin s \\ a\cos t \\ a \sin t \end{bmatrix}$ with $a > 0$.
(Euclidean or square torus)

Def. A surface ~~[illegible]~~ $S = \vec{X}(D)$ is differentiable if its coordinate fncs are

$$\vec{X}_s(s_0,t_0) = \frac{\partial \vec{X}}{\partial s}(s_0,t_0) = \begin{bmatrix} \frac{dx}{ds}(s_0,t_0) \\ \frac{dy}{ds}(s_0,t_0) \\ \frac{dz}{ds}(s_0,t_0) \end{bmatrix}$$

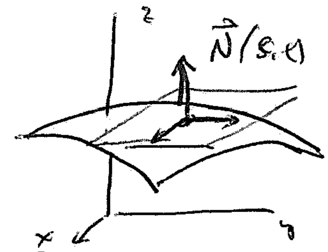
Same for $\vec{X}_t(s_0,t_0)$.

These are vectors tangent to the embedded surface $\vec{X}(D)$.

For $\vec{X}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ native $D\vec{X}_{3\times 2}$ $D\vec{X} = [\vec{X}_s, \vec{X}_t]$.

As $\vec{X}_s$ and $\vec{X}_t$ are always members of the tangent space to $\vec{X}(D) \in \mathbb{R}^3$, $\vec{X}_s \times \vec{X}_t$ is *normal* to the surface (when it is nonzero and C^1)

Call $\vec{N}(s_0,t_0) = \vec{X}_s \times \vec{X}_t\,(s_0,t_0)$



Def $S = \vec{X}(D)$ is called *smooth* @ $\vec{X}(s_0,t_0)$ if $\vec{X}$ is C^1 in a nbhd of (s_0,t_0) *and* if $\vec{N}(s_0,t_0) \neq \vec{0}$.

S is smooth if it is smooth everywhere.

Note: C^1 ensures no edges *only if* $\vec{N}(s_0,t_0) \neq \vec{0}$. This is just like for curves.

ex. For $\vec{X}(D) = S^2$,

$$\vec{X}_s = \begin{bmatrix} -a\sin s \sin t \\ a\cos s \sin t \\ 0 \end{bmatrix} = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

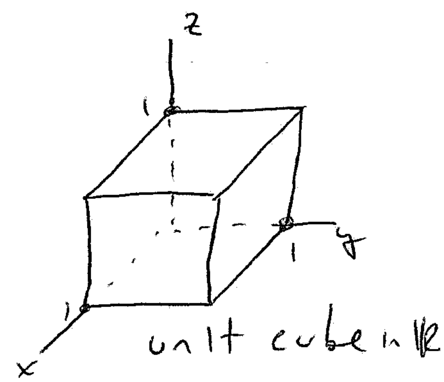
$$\vec{X}_t = \begin{bmatrix} a\cos s \cos t \\ a\sin s \cos t \\ -a\sin t \end{bmatrix}$$

So $\vec{N} = \vec{X}_s \times \vec{X}_t = -a\sin t \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Here $\vec{N}$ is nonzero except at $t = 0, \pi$ (except at the poles).

Now S^2 is smooth everywhere, even at the poles.
but not according to the parameterization!

"You cannot walk east or west @ the north pole!
You can only walk south!"

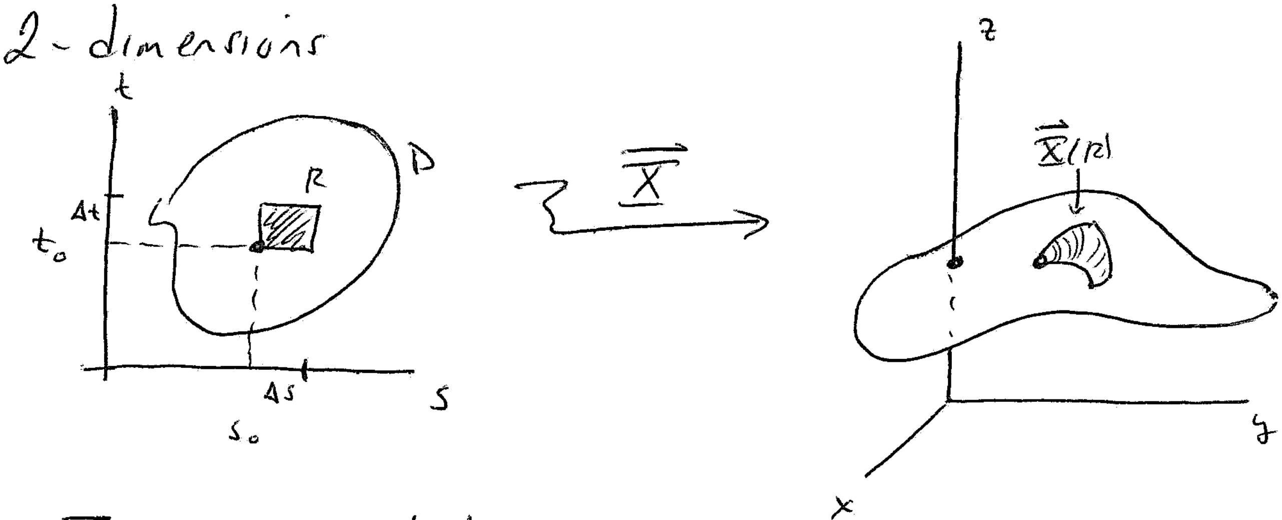
Def. A piecewise smooth parameterized surface is the union of images of finitely many parameterized surfaces $\vec{X}_i : D_i \to \mathbb{R}^3$, where each D_i is (1) elementary, (2) C^1 except possibly along ∂D_i, and (3) each $S_i = \vec{X}_i(D_i)$ is smooth except at possibly a finite # of pts.



The surface area of a parameterized surface.

Recall the length of $\vec{c} : [a,b] \to \mathbb{R}^n$ is $\int_a^b \|\vec{c}\,'(t)\| \, dt$

In 2-dimensions



Here $\overline{X}(R)$ may not be a rectangle, but can be approximated by one with sides

$$\vec{X}_s(s_0,t_0)\Delta s, \ \vec{X}_t(s_0,t_0)\Delta t.$$

And

$$\text{Area}(\vec{X}(R)) \approx \|\vec{X}_s(s_0,t_0)\Delta s \times \vec{X}_t(s_0,t_0)\Delta t\|$$

$$= \|\vec{X}_s(s_0,t_0) \times \vec{X}_t(s_0,t_0)\| \Delta s \Delta t$$

(Note: This is the area of the unit square in the tangent plane to $\vec{X}(D)$ at (s_0,t_0), scaled by Δs and Δt.)

In the limit, as $\Delta s, \Delta t \to 0$, we get $\|\vec{X}_s \times \vec{X}_t\| \, ds\, dt$

With the idea that $\text{area}(D) = \iint_D dA$, we get

$$\text{area}(S) = \iint_S dS = \iint_D \|\vec{X}_s \times \vec{X}_t\| \, ds\, dt$$

$$= \iint_D \|\vec{N}(s,t)\| \, ds\, dt$$

Here, $dS = \|\vec{N}(s,t)\| dA$ is the area form on the surface

Notes ① $dS = \|\vec{N}(s,t)\| dA$ is the 2-dimensional analogue to $ds = \|\vec{c}\,'(t)\| dt$ in the scalar-line integral.

② For $\underline{X}(s,t) = \big(x(s,t), y(s,t), z(s,t)\big) \in \mathbb{R}^3$,

$$\underline{X}_s \times \underline{X}_t = \begin{bmatrix} \frac{\partial(y,z)}{\partial(s,t)} \\ -\frac{\partial(x,z)}{\partial(s,t)} \\ \frac{\partial(x,y)}{\partial(s,t)} \end{bmatrix}, \text{ so}$$

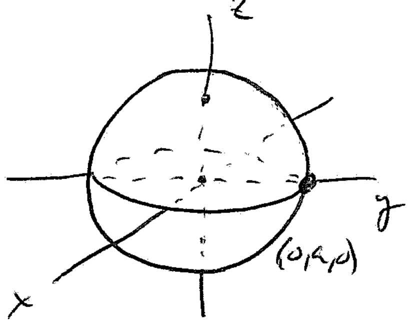
Here are Jacobian determinants

$$\text{area}(S) = \iint_D \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}\, ds\,dt$$

Compare this to arclength in Calc I: $\vec{c}(t) = (x(t), y(t))$,

$$\text{arclength}(\vec{c}) = \int_{\vec{c}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

ex. The surface area of a sphere of radius a?



Recall, $\mathbf{X}(s,t) = (a\cos s\sin t,\ a\sin s\sin t,\ a\cos t)$

So $\mathbf{X}_s(s,t) = \begin{bmatrix} -a\sin s\sin t \\ a\cos s\sin t \\ 0 \end{bmatrix}$, and

$\mathbf{X}_t(s,t) = \begin{bmatrix} a\cos s\cos t \\ a\sin s\cos t \\ -a\sin t \end{bmatrix}$

And $\mathbf{X}_s \times \mathbf{X}_t = \begin{bmatrix} \frac{\partial(y,z)}{\partial(s,t)} \\ -\frac{\partial(x,z)}{\partial(s,t)} \\ \frac{\partial(x,y)}{\partial(s,t)} \end{bmatrix}$, where

$$\boxed{\frac{\partial(y,z)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{vmatrix} = \begin{vmatrix} a\cos s\sin t & a\sin s\cos t \\ 0 & -a\sin t \end{vmatrix} = a^2\cos s\sin^2 t,}$$

$$\boxed{-\frac{\partial(x,z)}{\partial(s,t)} = -\begin{vmatrix} -a\sin s\sin t & a\cos s\cos t \\ 0 & -a\sin t \end{vmatrix} = -(a^2\sin s\sin^2 t)}$$

$$\boxed{\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} -a\sin s\sin t & a\cos s\cos t \\ a\cos s\sin t & a\sin s\cos t \end{vmatrix} = -a^2\sin^2 s\sin t\cos t - a^2\cos^2 s\sin t\cos t = -a^2\sin t\cos t}$$

Hence $\|\mathbf{X}_s \times \mathbf{X}_t\| = \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}$

$$= \sqrt{a^4\cos^2 s\sin^4 t + a^4\sin^2 s\sin^4 t + a^4\sin^2 t\cos^2 t} = a^2\sin t$$

$$\text{area}(S) = \int_0^{\pi}\int_0^{2\pi} \|\mathbf{X}_s \times \mathbf{X}_t\|\,ds\,dt = \int_0^{\pi}\int_0^{2\pi} a^2\sin t\,ds\,dt = \int_0^{\pi} 2\pi a^2\sin t\,dt$$

$$= -2\pi a^2\cos t\Big|_0^{\pi} = 2\pi a^2 + 2\pi a^2 = \boxed{4\pi a^2}$$

ex Let the surface be graph(f), for $f: D \subset \mathbb{R}^2 \to \mathbb{R}^3$

$\Rightarrow \Phi(D) = (x, y, f(x,y))$, for $f(x,y)$ on D.

Here $\Phi_x(x,y) = \begin{bmatrix} 1 \\ 0 \\ f_x \end{bmatrix}$, $\Phi_y(x,y) = \begin{bmatrix} 0 \\ 1 \\ f_y \end{bmatrix}$.

$$\Phi_x \times \Phi_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x \vec{i} - f_y \vec{j} + k = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$$

Then the surface area of $\Phi(D) = S$ is

$$\text{area}(\Phi(D)) = \iint_S dS = \iint_D \|\Phi_x \times \Phi_y\| \, dA = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA$$