Section 7.2

In the same way that we integrated functions (real-valued) and vector fields over curves, we can do so over surfaces.

(I) Real-valued (scalar) functions

- Like for scalar line integrals, if the surface in $\mathbb{R}^3$ is inside the domain of a real-valued function in $\mathbb{R}^3$, we can restrict the domain to the surface and integrate.
- If we parameterize the surface with coordinates on the surface, this is then a double integral.
- However, the resulting ~~integral~~ value should be parameter independent.
- Basically, we look to define

$$\iint_{\text{surface}} f \, dS, \text{ where } dS = \|N(s,t)\| \, ds \, dt$$

is a surface differential.

Def Let $\Sigma : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth param. surface, where D is bounded. Let f be a C^0 function on a domain that includes $\Sigma(D)$. Then the <u>scalar surface integral</u> of f along Σ is

$$\iint_{\Sigma} f\, dS = \iint_D f(\Sigma(s,t))\, \|\Sigma_s \times \Sigma_t\|\, ds\, dt$$

$$= \iint_D f(x(s,t), y(s,t), z(s,t)) \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}\, ds\, dt$$

<u>Notes</u>: ① Like for line integrals, dS is a scalar 2-form (ds is a scalar 1-form) and represents an infinitesimal change in ~~[illegible]~~ surface area along the surface.

② For $f(x,y,z) = 1$, this integral gives the surface area of $\Sigma(D)$.

③ In coordinates (s,t), this looks like a standard double integral.

④ If Σ is not smooth but has edges (piecewise smooth) then each smooth piece must be integrated separately and the results added together.

Def. Let $\mathbf{X}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth param. surface, where D is bounded. Let $\vec{F}$ be a C^0-vector field on a domain in $\mathbb{R}^3$ that includes $\mathbf{X}(D)$. Then the vector surface integral of $\vec{F}$ along $\mathbf{X}$ is

$$\iint_{\mathbf{X}} \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\mathbf{X}(s,t)) \cdot \vec{N}(s,t)\, ds\, dt$$

$$= \iint_D \vec{F}(x(s,t), y(s,t), z(s,t)) \cdot \begin{bmatrix} \frac{\partial(y,z)}{\partial(s,t)} \\ -\frac{\partial(x,z)}{\partial(s,t)} \\ \frac{\partial(x,y)}{\partial(s,t)} \end{bmatrix} ds\, dt.$$

Notes ① Here $d\vec{S} = \vec{N}(s,t)\,ds\,dt$ is a vector 2-form, the differential of surface area written in terms of the normal to the surface at (s,t).

② If we normalize the normal vector $\vec{n}(s,t) = \dfrac{\vec{N}(s,t)}{\|\vec{N}(s,t)\|}$, then

$$\int_{\mathbf{X}} \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\mathbf{X}(s,t)) \cdot \vec{N}(s,t)\, ds\, dt = \iint_D \vec{F}(\mathbf{X}(s,t)) \cdot \vec{n}(s,t)\, \|\vec{N}(s,t)\|\, ds\, dt$$

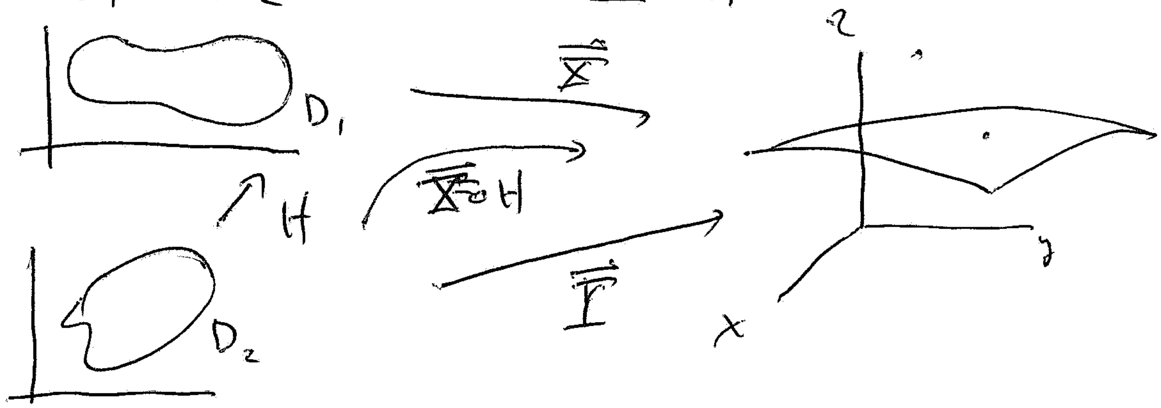
$$= \int_{\mathbf{X}} (\vec{F} \cdot \vec{n})\, dS$$

and the vector-surface integral of a vector field along a surface is also the scalar surface integral of the ~~~~ component of the vector field <u>normal</u> to the surface.

<u>This will be very important!</u>

(3) Reparameterizations of a surface play the same role as that of curves:

<u>Def.</u> Let $\vec{X}: D_1 \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\vec{Y}: D_2 \subset \mathbb{R}^2 \to \mathbb{R}^3$ be 2 parameterizations $\ni$ $\vec{X}(D_1) = \vec{Y}(D_2)$. $\vec{Y}$ is a reparameterization of $\vec{X}$ if ~~~~ $\exists$ a 1-1, onto $H: D_2 \to D_1$ with inverse $H^{-1}: D_1 \to D_2$ $\ni$ $\vec{Y} = \vec{X} \circ H$



The vector surface integral of a vector field equals the scalar surface integral of the normal component of the vector field to the surface.

<u>Interpretation</u> – $\iint_{\vec{X}} \vec{F} \cdot d\vec{S}$ measures the vector field flow <u>through</u> the surface. This is called the <u>flux</u> of $\vec{F}$ through $\vec{X}(D)$.

Compare this to $\int_{\vec{x}} \vec{F} \cdot d\vec{s}$, the circulation, the vector field flow <u>along</u> $\vec{x}(I)$.

<u>Other facts</u>

(I) Given a parameterization $\vec{X}: D_1 \subset \mathbb{R}^2 \to \mathbb{R}^3$ and a C^1, 1-1, onto $\vec{H}: D_2 \to D_1$ with inverse $H^{-1}: D_1 \to D_2$, a <u>reparameterization</u> of $\vec{X}$ is $\vec{Y}: D_2 \to \mathbb{R}^3$, where $\vec{Y} = \vec{X} \circ \vec{H}$.

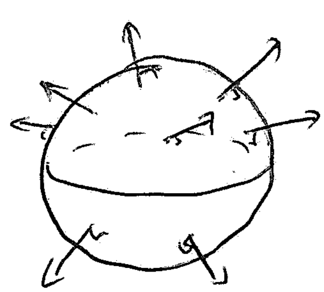
A reparameterization is called smooth if both $\vec{X}$ and $\vec{Y}$ are and if H is C^1.

II **Thm** For f C^0 on a domain including a smooth $\vec{X} : D \to \mathbb{R}^3$, then for any smooth reparameterization $\vec{Y}$ of $\vec{X}$,

$$\iint_{\vec{Y}} f \, dS = \iint_{\vec{X}} f \, dS$$

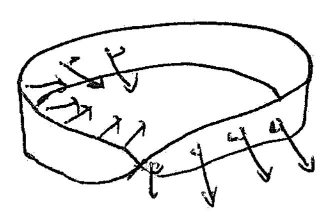
III For a curve, an orientation is a choice of continuously varying unit tangent vector along $\vec{c}$.

For a surface, an orientation is a choice of continuously varying unit normal vector along $\vec{X}$ (above vs. below, inside vs outside).



orientable

vs



non-orientable

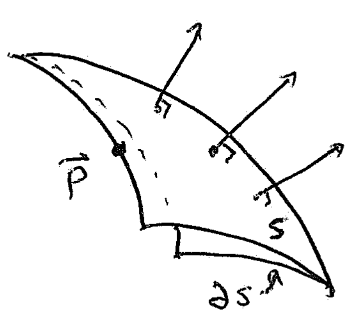
(IV) Thm: If a reparameterization $\underline{Y}$ preserves orientation (i.e., Jacobian$(\vec{H}) > 0$ everywhere)

$$\Longrightarrow \iint_{\vec{Y}} \vec{F} \cdot d\vec{S} = \iint_{\vec{X}} \vec{F} \cdot d\vec{S}$$

otherwise introduce a minus sign to RHS.

Note: Recall $\vec{N}(s,t) = \vec{\underline{X}}_s \times \vec{\underline{X}}_t = -\vec{\underline{X}}_t \times \vec{\underline{X}}_s$.

(V) Orienting a surface automatically orients boundary curves on that surface.



Let S be an oriented surface with boundary in $\mathbb{R}^3$ $\ni$ ∂S is a piecewise C^1 closed curve.

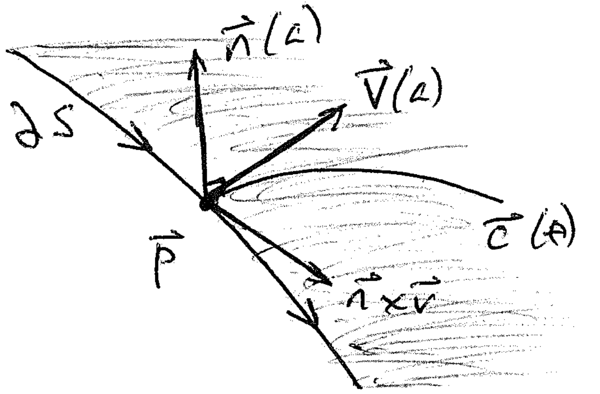
Let $\vec{p} \in \partial S$, where

$$\vec{p} = (s_0, t_0) = \big(x(s_0,t_0), y(s_0,t_0), z(s_0,t_0)\big)$$

and choose $\vec{c}: [a,b] \longrightarrow S \subset \mathbb{R}^3$ a smooth curve such that $\vec{c}(a) = \vec{p}$, and $\vec{c} \cap \partial S = \{\vec{p}\}$.

Define $\vec{n}(\vec{p}) = \lim_{t\to a} \vec{n}(\vec{c}(t))$, and

$\vec{v}(a) = \lim_{t\to a} \vec{c}\,'(t)$.



Here, $\vec{n}$ and $\vec{v}$ are based at $\vec{p}$ and are perpendicular. Hence they determine a 2-d subspace containing $\vec{v}$ and $\vec{n}$.

Then $\vec{n}\times\vec{v}$ is perpendicular to both and using the RHR determines a unique direction on ∂S.

This is the <u>direction</u> used in <u>Green's Thm</u>!