Section 7.3

Thm (Stoke's) Let S be a bounded, piece-wise smooth, oriented surface in $\mathbb{R}^3$, where ∂S consists of finitely many piecewise smooth closed curves oriented correctly. For $\vec{F}$ a C^1-vector field on a domain containing S,

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$$

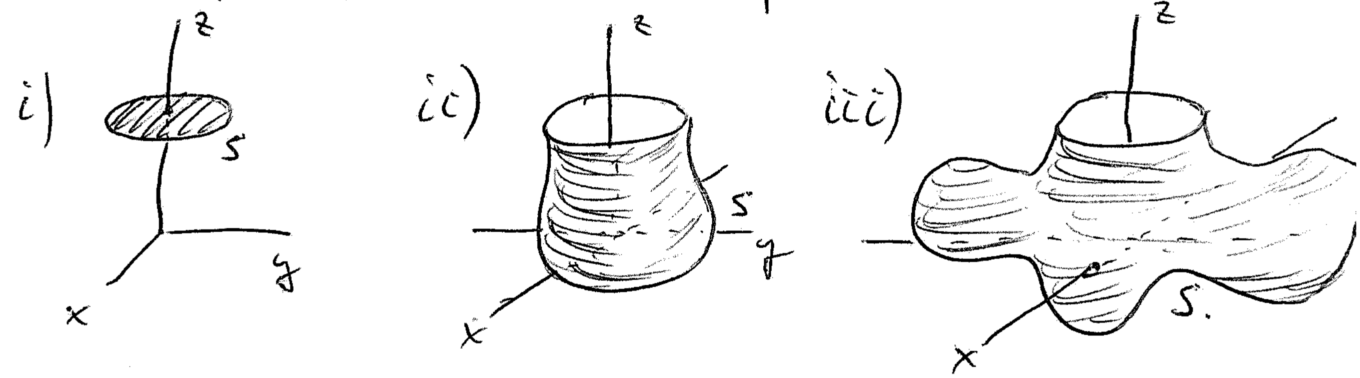
(the surface integral of the curl of $\vec{F}$ equals the line integral of $\vec{F}$ along ∂S).

Notes ① This is a lot like Green's Thm

i) LHS measures the normal component of the curl of $\vec{F}$ along S (amount of twisting in direction through S).

ii) RHS measures the circulation of $\vec{F}$ along ∂S

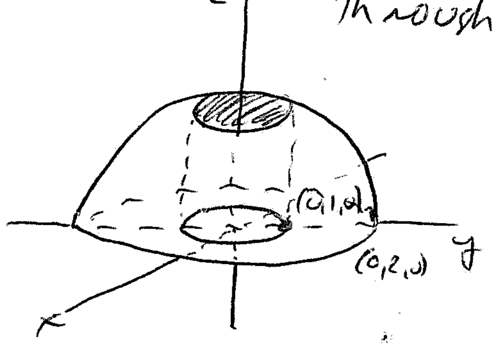
② It's almost as if the surface doesn't matter, only the boundary!



In each of these, $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$ is the same by Stoke's Thm.

③ Typical use – Sometimes the flux of the curl of $\vec{F}$ is hard to calculate across a bounded surface. But the ~~and~~ circulation along its boundary is not.

ex. Compute the flux of the curl of $\vec{F}(\vec{x}) = \begin{bmatrix} xz \\ yz \\ xy \end{bmatrix}$



through the surface of the sphere $x^2+y^2+z^2=4$ inside the cylinder $x^2+y^2=1$ above the xy-plane.

<u>Strategy</u> Since both the vector field and the surface satisfy Stokes' Thm ($\vec{F}$ is C^0 on $\mathbb{R}^3$, and S, oriented using the outward normal is orientable and bounded with a closed smooth boundary curve, oriented compatibly (counterclockwise wrt the outward normal or w/ S on the left walking upright on ∂S wrt the outward normal), we look to integrate the curl$(\vec{F})$ by instead calculating the circulation of F on ∂S.

<u>Solution</u> <u>Step 1</u>: Find eqn for ∂S

∂S is on both the sphere $x^2+y^2+z^2=4$ and the cylinder $x^2+y^2=1$. Hence $x^2+y^2+z^2 = 1+z^2 = 4$,

or $z^2 = 3$, or $z = \sqrt{3}$.

Parameterize ∂S via $\vec{c}: [0, 2\pi] \rightarrow \mathbb{R}^3$,

$$\vec{c}(t) = \begin{bmatrix} \cos t \\ \sin t \\ \sqrt{3} \end{bmatrix} \subset \mathbb{R}^3.$$

Step 2: Calculate the circulation of $\vec{F}$ on $\vec{c}$.

$$\iint_S \nabla \times \vec{F} \bullet d\vec{S} \overset{\text{Stokes Thm}}{=\!=} \int_{\partial S} \vec{F} \bullet d\vec{s} = \int_{\vec{c}} \vec{F}(\vec{c}(t)) \bullet \vec{c}\,'(t)\, dt$$

$$= \int_0^{2\pi} \begin{bmatrix} \sqrt{3}\cos t \\ \sqrt{3}\sin t \\ \cos t \sin t \end{bmatrix} \bullet \begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix} dt = \int_0^{2\pi} 0 \, dt = 0.$$

Suppose we did this calculation directly....

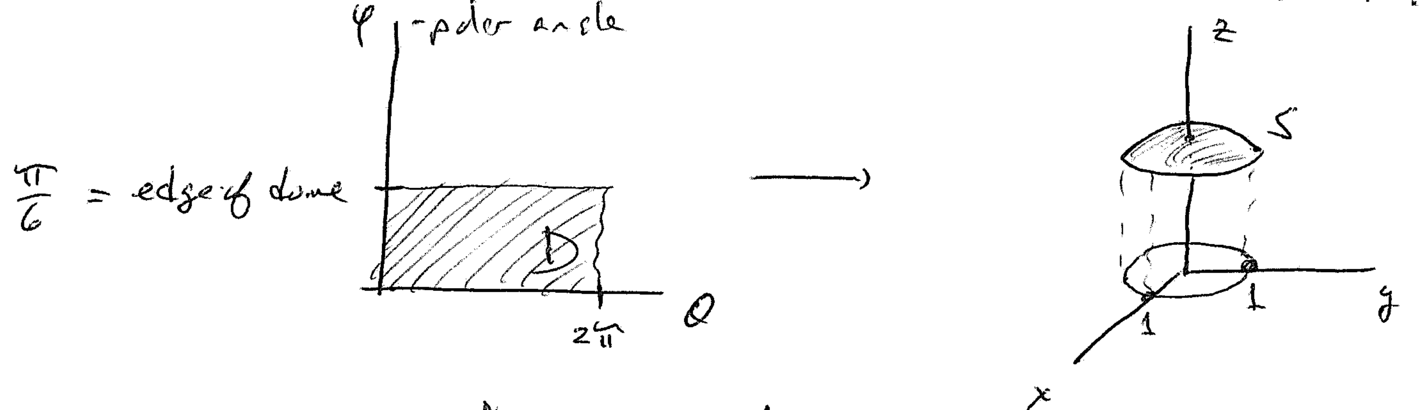
Strategy: Calculate curl($\vec{F}$). Then parameterize the surface S using spherical coordinates, noting that the parameterization domain is a rectangle. Do the double integral.

Solution: The curl($\vec{F}$) is easy enough:

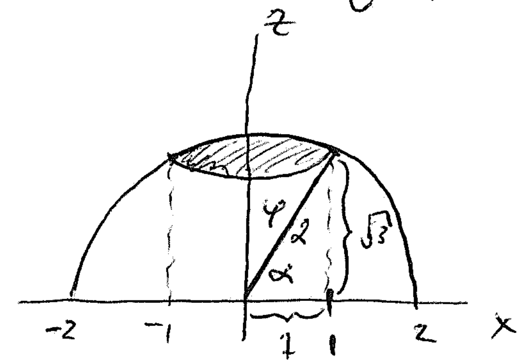
$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} = (x-y)\vec{i} - (y-x)\vec{j} + 0\vec{k}$$

$$= \begin{bmatrix} x-y \\ x-y \\ 0 \end{bmatrix}$$

Next, we parameterize S via spherical coords:



$$\vec{\Phi}(\theta,\varphi) = \begin{bmatrix} 2\cos\theta\sin\varphi \\ 2\sin\theta\sin\varphi \\ 2\cos\varphi \end{bmatrix}.$$ To find the polar angle corresponding to ∂S, use a cross section of the original drawing:



Here the triangle with interior angle α has short side 1 and hypotenuse 2.

Hence $\cos\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3} \Rightarrow \underline{\varphi = \frac{\pi}{6}}$.

Next, we ensure our idea of an orientation is correct. We want the outward pointing normal. Hence we want the unit normal vector on S to always have a positive z-component.

Here $\overline{X}_\theta = \begin{bmatrix} -2\sin\theta\sin\varphi \\ 2\cos\theta\sin\varphi \\ 0 \end{bmatrix}$, $\overline{X}_\varphi = \begin{bmatrix} 2\cos\theta\cos\varphi \\ 2\sin\theta\cos\varphi \\ -2\sin\varphi \end{bmatrix}$.

and $\vec{N} = \overline{X}_\theta \times \overline{X}_\varphi = \begin{bmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ \cos\varphi \end{bmatrix}(-4\sin\varphi)$.

So that $\vec{n} = \dfrac{\vec{N}}{\|\overline{X}_\theta \times \overline{X}_\varphi\|} = -\begin{bmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ \cos\varphi \end{bmatrix}$.

This points inward. This is fine but incompatible with our orientation of ∂S as counterclockwise.

Hence we re-orient S by simply <u>choosing</u> the other orientation: $\overline{X}_\varphi \times \overline{X}_\theta = -\overline{X}_\theta \times \overline{X}_\varphi$

We calculate $\displaystyle\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_D \nabla\times\vec{F}(\overline{X}(\theta,\varphi))\cdot(\overline{X}_\varphi\times\overline{X}_\theta)\, dA$

$$= \iint_D \left(\nabla\times\vec{F}(\overline{X}(\theta,\varphi))\cdot\vec{n}\right) dA$$

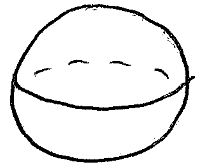
$$= \int_0^{\frac{\pi}{6}}\int_0^{2\pi} \begin{bmatrix} 2\sin\varphi(\cos\theta-\sin\theta) \\ 2\sin\varphi(\cos\theta-\sin\theta) \\ 0 \end{bmatrix}\cdot\begin{bmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ \cos\varphi \end{bmatrix} 4\sin\varphi\, d\theta\, d\varphi$$

$$= \int_0^{\frac{\pi}{6}} \int_0^{2\pi} 8 \sin^3\varphi \left(\cos^2\theta - \sin^2\theta\right) d\theta \, d\varphi$$

$$= \int_0^{\frac{\pi}{6}} 8 \sin^3\varphi \left(\int_0^{2\pi} \sin 2\theta \, d\theta \right) d\varphi .$$ But the inside integral is 0, since $\int_0^{2\pi} \sin 2\theta \, d\theta = \left(-\frac{1}{2}\cos 2\theta \Big|_0^{2\pi} \right) = 0$.

Hence the entire double integral is 0. ∎

Which was easier?

ex. A sphere  and a torus 

are closed surfaces.  is compact

but not closed (it has a boundary).

Creases and corners are OK.

By Stoke's Thm, the curl of __any__ vector field

over a closed surface is 0! (why?)

(II) Let $\vec{F}$ be a conservative vector field,

so that $\vec{F} = \nabla f$ for some potential f,

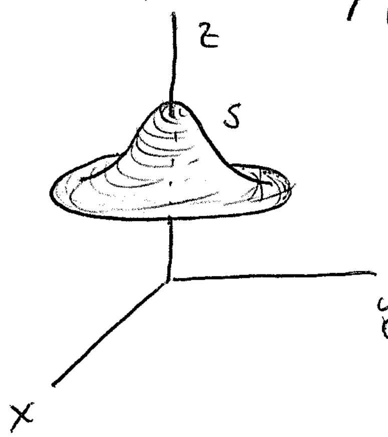
and S any surface that satisfies Stoke's

$$\Longrightarrow \oint_{\partial S} \vec{F} \cdot d\vec{s} = 0 \quad \left(\begin{array}{c}\text{no circulation} \\ \text{along } \partial S\end{array}\right).$$

__why?__ for any C^1-function f, $\nabla \times \nabla f = \vec{0}$.

~~ex.~~ (4) A nontypical use: Sometimes one can simply change the surface and leave the boundary unchanged.

ex. 7.3.2 pg. 492.



Calculate the flux of the curl of $\vec{F} = \begin{bmatrix} e^{y+z} - 2y \\ x e^{y+z} + y \\ e^{x+y} \end{bmatrix}$ on

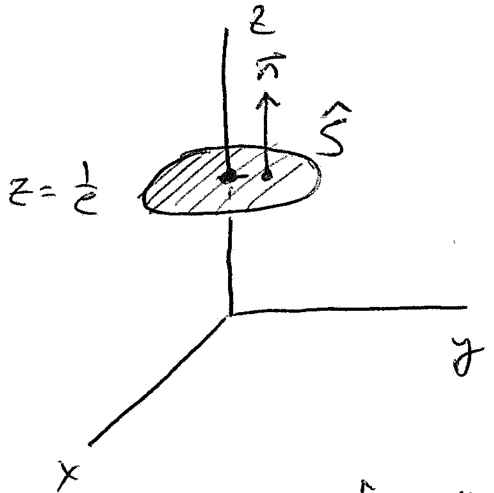
$$S = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, z \geq \tfrac{1}{e},\ z = e^{-(x^2+y^2)} \right\}$$

By the calculations in the book, one can use Stokes Thm here but both sides of the thm are very hard to integrate (see book).

However, by Stokes's Thm ANY surface with the same boundary will do:

Choose $\hat{S} = \left\{ (x,y,\tfrac{1}{e}) \in \mathbb{R}^3 \,\middle|\, x^2+y^2 \leq 1 \right\}$.

Here $\partial\hat{S} = \partial S = \left\{ (x,y,\tfrac{1}{e}) \in \mathbb{R}^3 \,\middle|\, x^2+y^2 = 1 \right\}$.



By Stoke's Thm,

$$\iint_S \nabla\times\vec F\cdot d\vec S = \iint_{\hat S} \nabla\times\vec F\cdot d\vec S$$

$$= \iint_{\hat S} \underbrace{(\nabla\times\vec F\cdot\vec n)}_{\text{curl of } \vec F \text{ in vertical direction}}\,dS$$

Here $\nabla\times\vec F = \begin{bmatrix} e^{x+y} - xe^{y+z} \\ e^{y+z} - e^{x+y} \\ 2 \end{bmatrix}$, $\vec n = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

So then $\nabla\times\vec F\cdot\vec n = 2\vec k$.

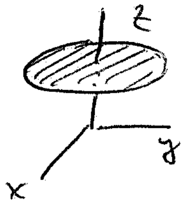
Thus $\iint_S \nabla\times\vec F\cdot d\vec S = \iint_{\hat S}(\nabla\times\vec F\cdot\vec n)\,dS = \iint_{\hat S} 2\,dS$

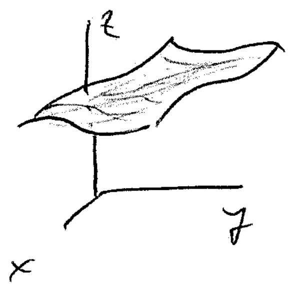
$$= 2\cdot(\text{area of } \hat S) = 2(\pi(1)^2) = \boxed{2\pi}$$

Some additional notes

(I) A surface is called <u>compact</u> if it is closed (includes all bdy pts) as a set, and bounded. It is called <u>closed</u> if it is compact without boundary.

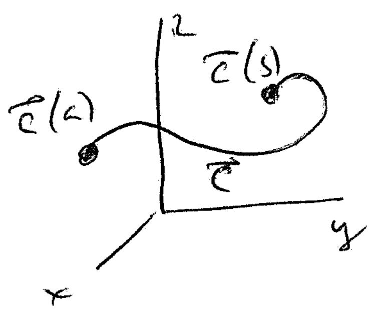
Ⓘ cont'd.

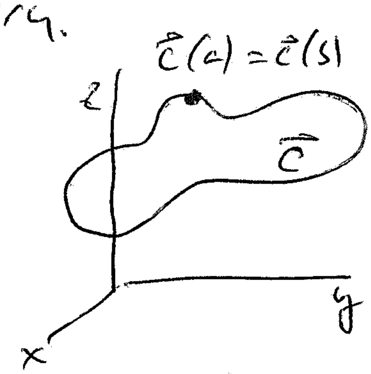
- Surfaces like the sphere  and the torus  are closed, while the disk 

and the carpet 

are compact but not closed (they have a boundary).

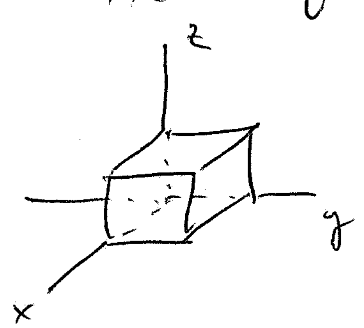
- Recall a curve in $\mathbb{R}^3$ is simple if it does not intersect itself. Hence a bounded simple curve with its end pts is compact. Its boundary is the end pts. A closed curve forms a loop and has no boundary.





$\partial\vec{c} = \{\vec{c}(a), \vec{c}(b)\}$ Here $\partial\vec{c} = \phi$

- Also the surface of the unit cube in $\mathbb{R}^3$ is closed. Creases and corners are fine but not considered boundary sts.



(II) By Stoke's Thm, the ~~surface~~ vector surface area (the flux) of the curl of any vector field over a closed surface is 0. Why?

Because $\oint_{\partial S} \vec{F} \cdot d\vec{s} = 0$ (Integrating over a null space ($\partial S = \phi$)).

(III) In contrast, let $\vec{F}$ be a conservative vector field, so $\vec{F} = \nabla f$ for some potential f. Then, if S is any surface that satisfies Stoke's, we have

$\oint_{\partial S} \vec{F} \cdot d\vec{s} = 0$ (The circulation of $\vec{F}$ along ∂S vanishes!)

Why is this? For a conservative vector field $\nabla \times \vec{F} = \nabla \times \nabla f = \vec{0}$. Hence

$$0 = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_S \vec{0} \cdot d\vec{S} \stackrel{St}{=} \oint \vec{F} \cdot d\vec{s}$$

And now we move up a dimension to Gauss' Thm.

__Thm__ (Gauss) Let D be a solid region in $\mathbb{R}^3$, with ∂D a finite set of piecewise smooth, closed, orientable surfaces (with orientation pointing away from D). For $\vec{F}$ a C^1 vector field defined on a domain containing D, we have

$$\left(\oiint_{\partial D} (\vec{F} \cdot \vec{n}) \cdot dS = \right) \oiint_{\partial D} \vec{F} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{F} \, dV \left(= \iiint_D (\operatorname{div} \vec{F}) \, dV \right)$$

Special Note:

In $\mathbb{R}^2$, any compact domain with nonempty interior has a set of closed curves as boundary.

In $\mathbb{R}^3$, any compact solid region w/ nonempty interior has a set of closed surfaces as boundary.

This generalizes to higher dimensions in the obvious way.

And leads to the conclusion:

The boundary of a compact region w/ nonempty interior has no boundary. Think about this.

The proof of Gauss is elementary and straightforward:

First, $\iiint_D (\operatorname{div} \vec{F})\, dV = \iiint_D \frac{\partial F_1}{\partial x}\, dV + \iiint_D \frac{\partial F_2}{\partial y}\, dV + \iiint \frac{\partial F_3}{\partial z}\, dV.$

Then $\iint_{\partial D} (\vec{F} \cdot \vec{n})\, dS = \iint_{\partial D} F_1 \vec{i} \cdot \vec{n}\, dS + \iint_{\partial D} F_2 \vec{j} \cdot \vec{n}\, dS + \iint_{\partial D} F_3 \vec{k} \cdot \vec{n}\, dS$

Show that each summand is equal to each, respectively.

exercise: Show $\iiint_D \frac{\partial F_1}{\partial x}\, dV = \iint_{\partial D} F_1 \vec{i} \cdot \vec{n}\, dS$. when

D is elementary in all directions.

We are now fully in a position to understand some concepts we previously only vaguely talked about:

- Divergence
- Curl.

(I) What is divergence?

Intuition: measures the infinitessimal expansion of volume under the flow of a vector field.

Actual: measures the aggregate flow across the boundary of an infinitessimal ball centered at a pt.

Thm Let $\vec{F}$ be a C^1 vector field in some nbhd of a pt $\vec{p} \in \mathbb{R}^3$. For $S_a = \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x} - \vec{p}\| = a\}$ the 2-sphere of radius $a>0$ centered at $\vec{p}$ and oriented outward,

$$\operatorname{div}(\vec{F}(\vec{p})) = \lim_{a \to 0^+} \frac{3}{4\pi a^3} \oiint_{S_a} \vec{F} \cdot d\vec{S}$$

proof: For any $f \in C^0[D \subset \mathbb{R}^3, \mathbb{R}]$, D a bounded solid region, $\exists\, \vec{q} \in \mathbb{R}^3 \ni$

$$\iiint_D f(x,y,z)\, dV = f(\vec{q})(\text{volume of } D)$$

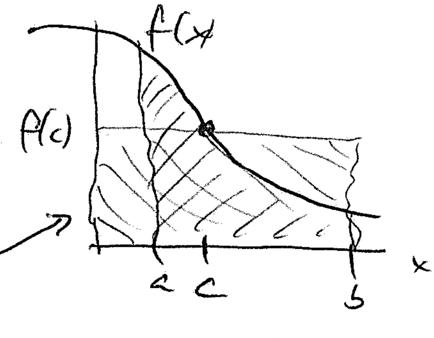
This is the Mean Value Thm for triple integrals.

Aside: Recall the MVT for integrals in Calc I:

For $f: I \to \mathbb{R}$ cont., $I = [a,b]$,

$$\int_a^b f(x)\,dx = \underbrace{f(c)\,(b-a)}_{\text{area of block}}$$

for some $c \in [a,b]$.



area is $\int_a^b f(x)\,dx$

Multiple integrals all have a similar MVT

proof cont'd. With $\vec{p}$ and S_a as in thm, we know, by the MVT for triple integrals, that $\exists\, \vec{q} \in B_a = \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x} - \vec{p}\| \le a\}$ $\ni$

$$\iiint_{B_a} (\operatorname{div} \vec{F})\,dV = \operatorname{div}(\vec{F}(\vec{q})) \cdot (\text{volume } B_a)$$
$$= \frac{4\pi a^3}{3} \cdot \operatorname{div}(\vec{F}(\vec{q}))$$

since $\operatorname{div} \vec{F}$ is just a real-valued fnc on B_a

With this, we can now use Gauss' Thm to establish the proof:

$$\lim_{a\to 0^+} \frac{3}{4\pi a^3} \oiint_{S_a} \vec{F}\cdot d\vec{S} \overset{\text{Gauss}}{=\!=} \lim_{a\to 0^+} \frac{3}{4\pi a^3} \iiint_{B_a} (\operatorname{div}\vec{F})\,dV$$

$$= \lim_{a\to 0^+} \frac{3}{4\pi a^3}\left(\frac{4\pi a^3}{3}\cdot \operatorname{div}\vec{F}(\vec{q})\right)$$

$$= \lim_{a\to 0^+} \operatorname{div}\vec{F}(\vec{q}) = \operatorname{div}\vec{F}(\vec{p}).$$

∎

So Gauss' Thm says that the amount of volume created or lost upon flowing along a vector field in D <u>equals</u> the total amount flowing through ∂D.

(II) What is curl?

<u>Intuition:</u> Curl measures the twisting effect of a vector field felt by flowing along it in $\mathbb{R}^3$.

<u>Actual:</u> measures the total circulation of a vector field along the edge of an infinitesimal disk normal to the vector field at a pt.

<u>Thm</u> For $\vec{F}$ a C^1 vector field in a nbhd of $\vec{p} \in \mathbb{R}^3$, ~~and~~ let $\vec{n}$ be a unit vector at $\vec{p}$ and $D_a = \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x} - \vec{p}\| \leq a, \vec{x} \cdot \vec{n} = 0\}$ be a disk of radius $a > 0$ centered at $\vec{p}$ and normal to $\vec{n}$. Orient D_a compatibly with $\vec{n}$ and also $C_a = \partial D_a$. Then

$$\operatorname{curl} \vec{F}(\vec{p}) \cdot \vec{n} = \lim_{a \to 0^+} \frac{1}{\pi a^2} \oint_{C_a} \vec{F} \cdot d\vec{s}$$

Hence curl at a pt is the infinitessimal circulation of $\vec{F}$ along a loop perpendicular to the direction of flow.

<u>proof</u> Exactly like the previous theorem but using Stokes' instead of Gauss.

<u>Notes</u> ① $\mathrm{div}(\vec{F}(\vec{p}))$ is also called the <u>flux density</u> of $\vec{F}$ at $\vec{p}$: It is the limit of flux per unit volume.

② $\mathrm{curl}(\vec{F}(\vec{p}))$ is also called the <u>circulation</u> <u>density</u> of $\vec{F}$ at $\vec{p}$: It is the circulation per unit area. of $\vec{F}$ at $\vec{p}$.

Stoke's Thm says that the total rotational effect of a vector field on a surface in $\mathbb{R}^3$ is equal to the aggregate push or hindrance of a particle on the edge.

Green's Thm is simply Stoke's Thm limited to domains in $\mathbb{R}^2$.

Example

ex. For $\vec{F} = \begin{bmatrix} 2x \\ y^2 \\ z^2 \end{bmatrix}$ and $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$

the unit sphere, find the flux of $\vec{F}$ through S

Solution Here the flux is $\iint_S (\vec{F} \cdot \vec{n})\, dS$. We use ~~Gauss~~ Gauss' Thm to instead ~~calculate~~ integrate $\operatorname{div}(\vec{F})$ on the unit ball in $\mathbb{R}^3$.

By Gauss, $\oiint_S (\vec{F} \cdot \vec{n})\, dS = \iiint_B (\operatorname{div} \vec{F})\, dV$

where $B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$.

Here $\iiint_B (\operatorname{div} \vec{F})\, dV = \iiint_B 2(1+y+z)\, dV$

$$= \underbrace{2\iiint_B dV}_{①} + \underbrace{2\iiint_B y\, dV}_{②} + \underbrace{2\iiint_B z\, dV}_{③}$$

For ②, $2\iiint_B y\, dV = 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} y\, dy\, dz\, dx$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \underbrace{\left(\frac{y^2}{2} \Big|_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} \right)}_{0} dz\, dx$$

$$= 0$$

Also for ③

So $$\iiint (\mathrm{div}\,\vec{F})\,dV = 2\iiint_B \underset{①}{d}V = 2(\mathrm{vol}\ B) = 2\left(\frac{4\pi(1^3)}{3}\right) = \frac{8\pi}{3}. \quad \blacksquare$$

Final note.

* The $\underbrace{\text{vector surface integral}}_{}$ (aggregate flux through surface or total ~~~~) of a divergence-free vector field on any closed surface is 0.

* Q: Is it true that if F satisfies $\oiint_S \vec{F}\cdot d\vec{S}$ $\forall$ closed surfaces S in a domain that $\mathrm{div}\,\vec{F} = 0$? I don't know.