Some (multi-) linear algebra

Let V be a n-dimensional vector space on $\mathbb{R}$.

Then pts $\vec{v} \in V$ are called vectors, where

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n, \quad v_i \in \mathbb{R} \quad \forall i = 1, \cdots, n.$$

A <u>linear functional</u> ((linear) 1-form, or covector) is a linear map $f: V \to \mathbb{R}$, where

$$f(c_1\vec{v} + c_2\vec{w}) = c_1 f(\vec{v}) + c_2 f(\vec{w})$$

$$\forall \vec{v}, \vec{w} \in V, \quad \forall c_1, c_2 \in \mathbb{R}.$$

The set of all <u>covectors</u> of V is again an n-dim vector space called the dual space to V, V^*.
(at least for finite dim V)

What is a basis for V^*? Let $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ith position

$i = 1, \cdots, n$ be the standard basis for V. Then, for each i, $\vec{e}_i^{\,*}: V \to \mathbb{R}$, $\vec{e}_i^{\,*}(\vec{w}) = w_i$ is the linear function which strips off the ith coord.

Thus $\{\vec{e}_1^*, \dots, \vec{e}_n^*\}$ form a basis for V^*, so that any covector (element of V^*) can be written as a linear combination of these:

$$\vec{v}^* = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \left(\sum_{i=1}^{n} \vec{e}_i^* \right) = v_1 \vec{e}_1^* + \cdots + v_n \vec{e}_n^*$$

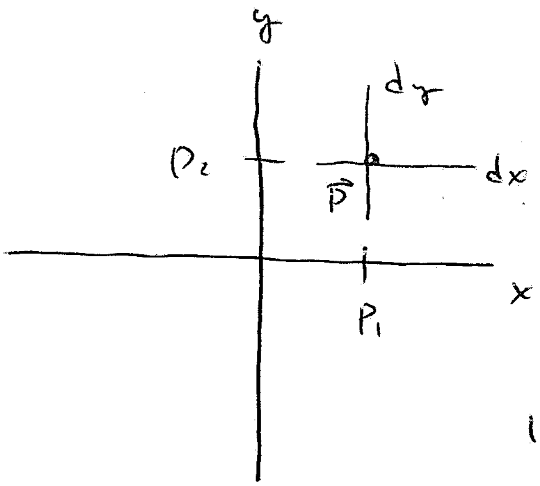
for $v_1, \dots, v_n \in \mathbb{R}$. Then $\vec{v}^* : \mathbb{R}^n \to \mathbb{R}$ is

$$\vec{v}^*(\vec{w}) = v_1 \vec{e}_1^*(\vec{w}) + \cdots + v_n \vec{e}_n^*(\vec{w})$$
$$= v_1 w_1 + \cdots + v_n w_n = \vec{v} \cdot \vec{w}$$
$$= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

<u>Notes</u> ① In this way, we often write linear functionals (covectors) as <u>row vectors</u>.

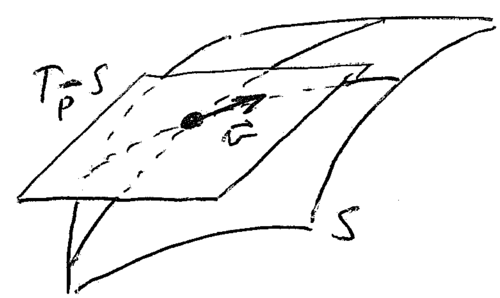
② The dot product $\mathrm{dot} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\mathrm{dot}(\vec{v}, \vec{w}) = \vec{v} \cdot \vec{w}$. is not a linear function. It is on each factor, and is called a <u>multilinear fnc</u>. The dot product w/ one slot filled is a linear functional.

③ In $\mathbb{R}^3$, each $\vec{p} \in \mathbb{R}^3$ has a tangent space $T_{\vec{p}}\mathbb{R}^3$, another copy of $\mathbb{R}^3$, but with its origin at $\vec{p}$. It is a <u>different space</u>!!!



For coordinates $(x_1, \dots, x_n)$ on $\mathbb{R}^n$, define a coordinate system on $T_{\vec p}\mathbb{R}^n$ as $(dx_1, \dots, dx_n)$, where dx_i is the infinitesimal change in x_i at $\vec p \in \mathbb{R}^n$. Here, each dx_i is a linear functional on $T_{\vec p}\mathbb{R}^n$ since for $\vec v \in T_{\vec p}\mathbb{R}^n$, $dx_i(\vec v) = v_i$.

Notes ① Think of a parameterized surface in $\mathbb{R}^n$, and it is easier to see how $\vec v \in T_{\vec p}S$ but $\vec v \notin S$.



② This definition of dx_i works because coordinates are actually linear functionals on a space (at least Cartesian ones), projections onto the factors of the space, which are linear functions.

Let $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in \mathbb{R}^2$. Then $x : \mathbb{R}^2 \to \mathbb{R}$, $y : \mathbb{R}^2 \to \mathbb{R}$

can be defined as $x(\vec{p}) = p_1$, $y(\vec{p}) = p_2$

These coordinate functions are linear and hence differentiable, and

$$Dx_{\vec{p}} : T_{\vec{p}}\mathbb{R}^2 \to \mathbb{R} \qquad Dx_{\vec{p}} = [1 \;\; 0]$$
$$Dy_{\vec{p}} : T_{\vec{p}}\mathbb{R}^2 \to \mathbb{R} \qquad Dy_{\vec{p}} = [0 \;\; 1].$$

Given a $\vec{v} \in T_{\vec{p}}\mathbb{R}^2$, $Dx_{\vec{p}}(\vec{v}) = v_1$, $Dy_{\vec{p}}(\vec{v}) = v_2$

Use this to define coordinates directly on $T_{\vec{p}}\mathbb{R}^2$, (dx, dy) where
$$dx = Dx_{\vec{p}} = [1 \;\; 0]$$
$$dy = Dy_{\vec{p}} = [0 \;\; 1].$$

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ex Let $\vec{v} \in \mathbb{R}^3$ so that $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$.

Then $x : \mathbb{R}^3 \to \mathbb{R}$ $\quad x(\vec{v}) = \vec{v} \cdot \vec{i} = v_1$

or $y : \mathbb{R}^3 \to \mathbb{R}$ $\quad y(\vec{v}) = \vec{v} \cdot \vec{e}_2 = \vec{e}_2^{\,*}(\vec{v}) = v_2$.

Either way, we often "abuse notation" for convenience and say $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$.

Geometrically, a linear functional on $\mathbb{R}^n$ looks like

$$\omega = a_1\,dx_1 + \cdots + a_n\,dx_n = \vec{a}\cdot d\vec{x}$$

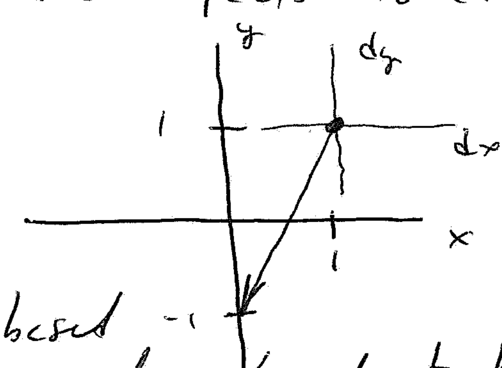
where $\vec{a}$ is the coefficient vector, and $d\vec{x} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$ is the basis of coefficient covectors in $\mathbb{R}^n$.

ex. Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -4 \\ -5 \\ -6 \end{bmatrix} \in \mathbb{R}^3$. Then

$$\begin{aligned} \omega(\vec{v}) &= a_1\,dx(\vec{v}) + a_2\,dy(\vec{v}) + a_3\,dz(\vec{v}) \\ &= a_1 v_1 + a_2 v_2 + a_3 v_3 = \vec{a}\cdot\vec{v} \\ &= \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 1(-4) + 2(-5) + 3(-6) = -32 \end{aligned}$$

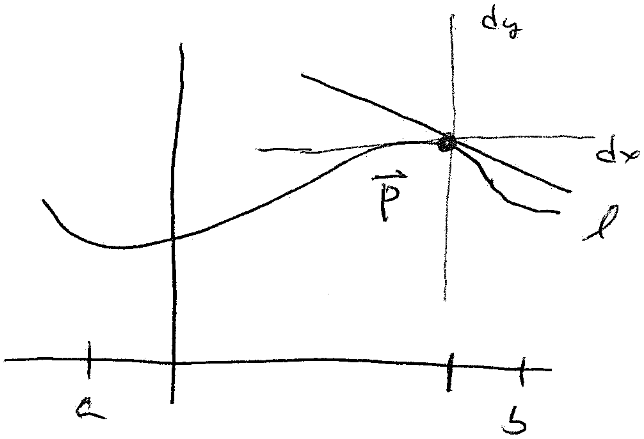
ex. Also, keep in mind where objects live!

Let $\vec{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \in T_{[1,1]}\mathbb{R}^2$.



Then, while we envision $\vec{v}$ as a vector in $\mathbb{R}^2$ based at $[1,1]$ it is really a vector based at the origin of $T_{\vec{p}}\mathbb{R}^2$ where $\vec{p} = [1,1]$.

ex. Let $\vec{c}:[a,b]\longrightarrow\mathbb{R}^2$ be a C^1 curve:

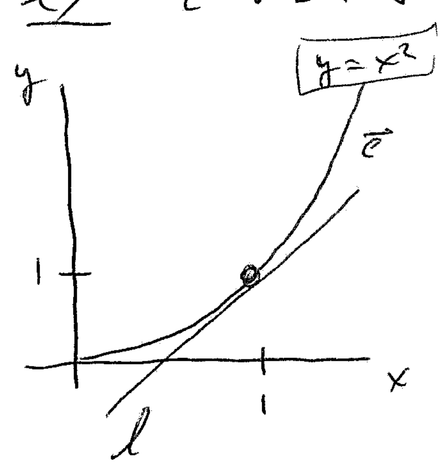


Since, for $\vec{p}\in\mathbb{R}^2$, $T_{\vec{p}}\mathbb{R}^2$ is not the same plane as $\mathbb{R}^2$ (it has different coordinates),

we can write the tangent line l via the coordinates of $T_{\vec{p}}\mathbb{R}^2$, since l is the set of all tangent vectors to $\vec{c}$ at $\vec{p}$, $l\subset T_{\vec{p}}\mathbb{R}^2$ (and not really $l\subset\mathbb{R}^2$):

eqn for l in $T_{\vec{p}}\mathbb{R}^2$: $dy=(\text{const})\,dx$

ex $\vec{c}:[0,2]\longrightarrow\mathbb{R}^2$, $\vec{c}(t)=(t,t^2)$



In the xy-plane, the eqn for l at $\vec{p}=\begin{bmatrix}1\\1\end{bmatrix}$ is

$$(y-1)=2(x-1)$$
$$y=2x-1$$

However, in $T_{\vec{p}}\mathbb{R}^2$, the eqn for l is

$dy=2\,dx$, or $\boxed{\dfrac{dy}{dx}=2}$

• at $\vec{p}=\begin{bmatrix}3\\9\end{bmatrix}$,
$dy=6\,dx$

Def A one form on a region $D \subset \mathbb{R}^n$ is a choice of a linear one form on each tangent space to D which varies continuously wrt $\vec{p} \in D$.

Note ① Sounds a lot like a vector-field, which is a choice of vector in each tangent space to D which varies cont. wrt $\vec{p} \in D$.

② But instead of a choice of vector, a one form is a choice of a covector, or linear functional on each tangent space.

a 1-form on D is a covector field on D!

ex. On $\mathbb{R}$, a generic 1-form looks like $f(x)\,dx$ for f a C^0-function on $\mathbb{R}$. So for $v \in T_x\mathbb{R}$

$$\omega(\cancel{x}) = f(\cancel{x})\,dx = a\,dx \qquad \text{i.e.}$$
$$\omega(v) = f(x)\,dx(v) = a\,dx(v)$$

where $a = f(\cancel{x}) \in \mathbb{R}$.

On $\mathbb{R}^n$, $\omega = f_1(\vec{x})\,dx_1 + \cdots + f_n(\vec{x})\,dx_n = \sum_{i=1}^{n} f_i(\vec{x})\,dx_i$

$$= \vec{F} \cdot d\vec{x}$$

where $\vec{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$, and $d\vec{x} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$

recognize this from vector line integral $d\vec{s} = \begin{bmatrix} dx_1 \\ \vdots \end{bmatrix}$?

Notes ① This $d\vec{x}$ <u>is</u> the $d\vec{s}$ in the definition of the vector line integral $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$.

② In a sense, integrating a vector field along a curve <u>is</u> adding (integrating) a 1-form along the curve.

③ A 1-form $\omega = \sum_{i=1}^{n} f_i(\vec{x})\,dx_i$ on $D \subset \mathbb{R}^n$ is a differential one-form if $f_i \in C^1 \ \forall i$.

④ For <u>any</u> real-valued $f: D \subset \mathbb{R}^n \to \mathbb{R}$, its differential $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$ is a diff. 1-form. But 1-forms do not have to come from functions.

⑤ Just as a vector field is a field of vectors (one for each pt in a space), and a function is a field of scalars, a diff. 1-form is a field of covectors, or linear functionals (one ~~for each~~ on each $T_{\bar{p}}D$, for $\bar{p} \in D$.)

Since dx and dy are linear functionals on $\mathbb{R}^2$ (coordinates on $T_{\vec{p}}\mathbb{R}^2$), they are covectors. One can multiply them together but, like vectors, the product is not always like the factors

(inner, outer, ~~dot~~ cross are all products of vectors with different output structure).

Features of any product of 1 forms

① Multiplication needs to be closed.

② Need the product to be linear, ~~at~~ ~~least~~ on each factor. (multilinear)

③ Can potentially act on pairs of vectors

Def The wedge product of 2 linear oneforms on $\mathbb{R}^n$ is

$$\omega \wedge \nu(\vec{v}_1, \vec{v}_2) = \begin{vmatrix} \omega(\vec{v}_1) & \omega(\vec{v}_2) \\ \nu(\vec{v}_1) & \nu(\vec{v}_2) \end{vmatrix}$$

$$= \omega(\vec{v}_1)\,\nu(\vec{v}_2) - \omega(\vec{v}_2)\,\nu(\vec{v}_1)$$

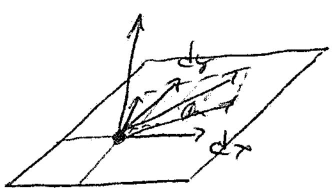
Here $\omega \wedge \nu : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is linear on each factor, but is not linear.
(this is another example of a multilinear func).

<u>Interpretation</u> - Think of a plane ~~whose~~ with coordinate axes

$$\hat{v}_1 = \begin{bmatrix} \omega(\vec{v}_1) \\ \nu(\vec{v}_1) \end{bmatrix}$$

$$\hat{v}_2 = \begin{bmatrix} \omega(\vec{v}_2) \\ \nu(\vec{v}_2) \end{bmatrix}$$

$\Longrightarrow$ $\omega \wedge \nu(\vec{v}_1, \vec{v}_2) = (\hat{v}_1 \times \hat{v}_2) \cdot \vec{k}$
is the <u>signed</u> <u>area</u> of the parallelogram whose sides are $\hat{v}_1, \hat{v}_2$.



<u>Properties</u>
1. $\nu \times \omega(\vec{v}_1, \vec{v}_2) = -\omega \wedge \nu(\vec{v}_1, \vec{v}_2)$
   Forms are skew-symmetric
2. $\omega \wedge \nu(\vec{v}_1, \vec{v}_1) = 0$ always
3. $(\omega + \nu) \wedge \mu = \omega \wedge \mu + \nu \wedge \mu$
4. $\omega \wedge \omega(\vec{v}_1, \vec{v}_2) = 0$ always

For ω, ν 1-forms on $\mathbb{R}^n$, $\omega \wedge \nu$ is called a 2-form on $\mathbb{R}^n$, where
$\omega \wedge \nu : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ (acts on pairs of vectors).

Let $\omega = \sum a_i dx_i$, $\nu = \sum b_j dx_j$

be 2 linear 1-forms

$$\Rightarrow \omega \wedge \nu = \sum_{i,j=1}^{n} a_i dx_i \wedge b_j dx_j$$

$$= \sum_{i,j=1} a_i b_j \, dx_i \wedge dx_j$$

as forms are linear on each factor.

However, when $i = j$, $dx_i \wedge dx_j = 0$, and in general $dx_i \wedge dx_j = - dx_j \wedge dx_i$.

ex. In $\mathbb{R}^3$, with coordinates x, y, z, let

$$\omega = a_1 dx + a_2 dy + a_3 dz$$
$$\nu = b_1 dx + b_2 dy + b_3 dz$$

Then

$$\begin{aligned}
\omega \wedge \nu &= a_1 b_1 \, dx \wedge dx + a_1 b_2 \, dx \wedge dy + a_1 b_3 \, dx \wedge dz \\
&\quad + a_2 b_1 \, dy \wedge dx + a_2 b_2 \, dy \wedge dy + a_2 b_3 \, dy \wedge dz \\
&\quad + a_3 b_1 \, dz \wedge dx + a_3 b_2 \, dz \wedge dy + a_3 b_3 \, dz \wedge dz \\
&= (a_2 b_3 - a_3 b_2) \, dy \wedge dz \\
&\quad + (a_3 b_1 - a_1 b_3) \, dz \wedge dx \\
&\quad + (a_1 b_2 - a_2 b_1) \, dx \wedge dy
\end{aligned}$$

~~Def~~ ~~exp.~~ A differentiable 2-form on $\mathbb{R}^n$ (a region in $\mathbb{R}^n$) is just a choice of a linear 2-form on each tangent space to $\mathbb{R}^n$.

For $\omega = \sum_{i=1}^{n} f_i(\vec{x})\, dx_i$, $\nu = \sum_{j=1}^{n} g_j(\vec{x})\, dx_j$

we have $\omega \wedge \nu = \sum_{i,j=1}^{n} f_i(\vec{x}) g_j(\vec{x})\, dx_i \wedge dx_j$

with all appropriate cancellations and skew symmetries

ex. let $\omega = x^2 y\, dx \wedge dy - xz\, dy \wedge dz$

be a 2-form on $\mathbb{R}^3$, and $\vec{p} = \begin{bmatrix}1\\2\\3\end{bmatrix} \in \mathbb{R}^3$

$$\Rightarrow \omega_{\vec{p}} = x^2 y\Big|_{\left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]} dx \wedge dy - xz\Big|_{\left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]} dy \wedge dz$$

$\omega_{\vec{p}} = 2\, dx \wedge dy - 3\, dy \wedge dz$, a linear 2-form on $T_{\vec{p}}\mathbb{R}^3$.

~~Choosing~~ Choose 2 vectors in $T_{\vec{p}}\mathbb{R}^3$, $\vec{v}_1 = \begin{bmatrix}4\\0\\-1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}3\\1\\2\end{bmatrix}$

$$\Rightarrow \omega_{\vec{p}}(\vec{v}_1, \vec{v}_2) = 2\, dx \wedge dy - 3\, dy \wedge dz \left(\begin{bmatrix}4\\0\\-1\end{bmatrix}, \begin{bmatrix}3\\1\\2\end{bmatrix} \right)$$

$$= 2 \begin{vmatrix} 4 & 3 \\ 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 2(4) - 3(1) = 5.$$

where $\begin{vmatrix} 4 & 3 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} dx\begin{bmatrix}4\\0\\-1\end{bmatrix} & dx\begin{bmatrix}3\\1\\2\end{bmatrix} \\ dy\begin{bmatrix}4\\0\\-1\end{bmatrix} & dy\begin{bmatrix}3\\1\\2\end{bmatrix} \end{vmatrix}$.

Thus, defined on $\mathbb{R}^3$, w/ coordinates x, y, z,

a general 2-form looks like

$$\eta = f_1(x,y,z)\, dx \wedge dy + f_2(x,y,z)\, dx \wedge dz + f_3(x,y,z)\, dy \wedge dz$$

And if η is the wedge of 2 1-forms

$$\omega = g_1\, dx + g_2\, dy + g_3\, dz$$

$$\vartheta = h_1\, dx + h_2\, dy + h_3\, dz$$

Then $f_1(x,y,z) = g_1(x,y,z)\, h_2(x,y,z) - g_2(x,y,z)\, h_1(x,y,z)$

and so on.

Notes ~~A generic way to write a differential 2-form in $\mathbb{R}$~~

① Completely generalizes to $\mathbb{R}^n$, $n > 3$ with similar structure

② A generic way to write a diff 2-form on $\mathbb{R}^n$ w/ coords $x_1, \dots, x_n$ is

$$\eta = \sum_{i,j=1}^{n} f_{i,j}\, dx_i \wedge dx_j$$

with all appropriate cancellations and simplifications.

(3) ~~Def~~. We can continue to construct higher order forms by the wedge product:

(4) Let $\omega_i = \sum f_i \, dx_i$ be a set of diff 1 one forms on $\mathbb{R}^m$, $i = 1, \dots, n$.

$\Longrightarrow$ $\eta = \omega_1 \wedge \cdots \wedge \omega_n$ is a differential n-form on $\mathbb{R}^m$ which will ultimately look like
$$\eta = \sum F_{i_1 \cdots i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$
with a lot of vanishings and simplifications.

Notes: (1)

At a pt $\vec{p} \in \mathbb{R}^m$, $\eta_{\vec{p}} : \underbrace{T_{\vec{p}}\mathbb{R}^m \times \cdots \times T_{\vec{p}}\mathbb{R}^m}_{n \text{ factors}} \to \mathbb{R}$,

$$\eta_{\vec{p}}(\vec{v}_1, \dots, \vec{v}_n) = \begin{vmatrix} \omega_1(\vec{v}_1) & \cdots & \omega_1(\vec{v}_n) \\ \vdots & & \\ \omega_n(\vec{v}_1) & \cdots & \omega_n(\vec{v}_n) \end{vmatrix}$$

This is a very mechanical process.

(2) ~~Note~~ also that $\eta_{\vec{p}}$ is linear in each factor:

$$\eta_{\vec{p}}(\vec{v}_1, \dots, \vec{v}_{i-1}, c_1\vec{u}_1 + c_2\vec{u}_2, \vec{v}_{i+1}, \dots, \vec{v}_n) =$$
$$c_1 \eta_{\vec{p}}(\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{u}_1, \vec{v}_{i+1}, \dots, \vec{v}_n) + c_2 \eta_{\vec{p}}(\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{u}_2, \vec{v}_{i+1}, \dots, \vec{v}_n)$$

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(3) (3) For $\omega = \sum F_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ a diff k-form

and $\nu = \sum G_{j_1 \cdots j_l} dx_{j_1} \wedge \cdots \wedge dx_{j_l}$ a diff l-form

$$\omega \wedge \nu = \sum F_{i_1 \cdots i_k} G_{j_1 \cdots j_l} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}$$

is a diff k+l form

(4) Call a C^∞ function on $\mathbb{R}^n$ a diff 0-form.
Then for ω a k-form,

$$f \wedge \omega = f\left(\sum F_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)$$
$$= \sum f \cdot F_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k})$$

is still a ~~k-form~~ 0+k = k-form.

(5) ∃ 2 forms which do not ~~are~~ arise as wedge products of 1-forms

(6) A n-form on $\mathbb{R}^n$ is also called a <u>volume form</u>.

(7) The wedge product is also called the
<u>exterior</u> product: The product of 2 n-forms is not an n-form (except when n=0).

Def. A differential n-form on $\mathbb{R}^m$, $m \geq n$,

$$\eta = \sum_{i_1, \cdots, i_n = 1}^{m} F_{i_1 \cdots i_n}\, dx_{i_1} \wedge \cdots \wedge dx_{i_n}$$

is a continuous family of linear n-forms $\eta_{\vec{p}}$, parameterized by $\vec{p} \in \mathbb{R}^m$, such that at each $\vec{p} \in \mathbb{R}^m$,

$$\eta_{\vec{p}} : T_{\vec{p}}\mathbb{R}^m \times \cdots \times T_{\vec{p}}\mathbb{R}^m \longrightarrow \mathbb{R}$$

is linear on each factor.

Alternate View

For each $\vec{p} \in \mathbb{R}^m$, and each factor $T_{\vec{p}}\mathbb{R}^m$ of $\eta_{\vec{p}}$, a choice of $\vec{v}_{\vec{p}} \in T_{\vec{p}}\mathbb{R}^m$ is a vector field on the region in $\mathbb{R}^m$.

Hence a differential n-form on a region in $\mathbb{R}^m$ "acts" on n-vector fields of $\mathbb{R}^m$ simultaneously and returns a function on the region.

ex. Let $\vec{F} = \begin{bmatrix} 2y \\ 0 \\ -x \end{bmatrix}$ be a vector field on $\mathbb{R}^3$,

and $\omega = x^2 y\,dx - x\,dy + y^2 z\,dz$

be a differential 1-form.

Then at $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\vec{v} \in \vec{F}$,

$$\begin{aligned}
\omega(\vec{v}) &= (x^2 y\,dx - x\,dy + y^2 z\,dz)(\vec{v}) \\
&= x^2 y\,dx(\vec{v}) - x\,dy(\vec{v}) + y^2 z\,dz(\vec{v}) \\
&= x^2 y(2y) - x(0) + y^2 z(-x) \\
&= 2x^2 y^2 - xy^2 z
\end{aligned}$$

Hence $\omega(\vec{F}) : \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$$\omega(\vec{F})(x, y, z) = 2x^2 y^2 - xy^2 z$$

Another view: forms are generalized integrands. One can "add" them up on each tangent space over an appropriate domain.

① Let $f: \mathbb{R} \to \mathbb{R}$ be C^0. Then $\omega = f(x)\,dx$ is a continuous 1-form. For $R = [a,b]$ an interval,

$$\int_R \omega = \int_a^b f(x)\,dx.$$

② For $\omega = F_1(\vec{x})\,dx_1 + \cdots + F_n(\vec{x})\,dx_n$ a diff. 1-form on $\mathbb{R}^n$, define

$$\int_{\vec{c}} \omega = \int_{\vec{c}} \vec{F} \cdot d\vec{s}$$

where $\vec{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$ is a C^1-vector field, and

$d\vec{s} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$ and $\vec{c}$ is some C^1-curve in $\mathbb{R}^n$.

This is just the vector line integral of $\vec{F}$ over $\vec{c}$.

③ If $\omega = f(x,y)\,dx \wedge dy$, $R \subset \mathbb{R}^2$ some region,

$$\int_R \omega = \iint_R f(x,y)\,dx\,dy$$

<u>Curious Fact</u>: Recall that $dy \wedge dx = -dx \wedge dy$.

Hence $-\omega = f(x,y)\, dy \wedge dx$.

Let $R \subset \mathbb{R}^2$ be a rectangular region. Then, by Fubini's Thm

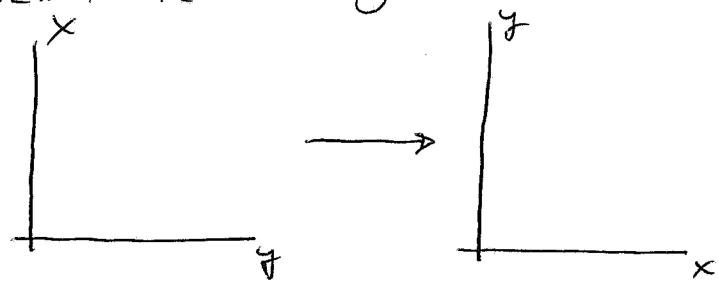
$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx, \quad \text{but}$$

$$\|$$

$$\int_R \omega = \iint_R f(x,y)\, dx \wedge dy \overset{?}{=} \iint_R f(x,y)\, dy \wedge dx = -\int_R \omega$$

What is wrong here? Actually nothing! Switching the order of integration is like a reparameterization of plane:

$$T(y,x) = (x,y)$$



which changes orientation

since $\mathrm{Jac}(T) = \left| \frac{\partial(y,x)}{\partial(x,y)} \right|$

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

The standard Change of variables formula is

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y) \left| \frac{\partial(x,y)}{\partial(y,x)} \right| dy\,dx$$

with <u>absolute</u> <u>value</u> of Jacobian!

But this absolute value is artificial, and a convenient shortcut to mark a deeper structure.

In Fubini's Thm, we avoid orientation and forms..

With forms, orientation is critical!

- The change in orientation introduces a minus sign.
- But the switch from $dx \wedge dy$ to $dy \wedge dx$ cancels it out.

In the iterated integral form of Fubini, both sign changes are hidden.

---

(4) Let $D \subseteq \mathbb{R}^2$ be the domain of $\vec{X} : D \to \mathbb{R}^3$.

Then, for ω a diff. 2-form on $\mathbb{R}^3$,

$\omega|_{\vec{X}(D)}$ is a 2-form on $\vec{X}(D)$ which

(1) can be expressed in the parameters,

(2) can be integrated in the parameters.

We set

$$\int_{\vec{X}} \omega = \int_D \omega_{\vec{X}(s,t)}\left(\vec{X}_s(s,t), \vec{X}_t(s,t)\right) ds \wedge dt$$

$$= \int_{\vec{X}} \vec{F} \cdot d\vec{S}, \text{ for } \vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}, \; d\vec{S} = \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix}$$

and $\omega = F_1(\vec{x})\, dy \wedge dz + F_2(\vec{x})\, dz \wedge dx + F_3(\vec{x})\, dx \wedge dy$.

④ Let $R \subset \mathbb{R}^m$ be an n-dimensional region, parameterized by $\Phi : D \to \mathbb{R}^m$, $\Phi(D) = R$.

Then, for ω a differential n-form on $\mathbb{R}^m$,

$$\omega\big|_{\Phi(D)} = \omega_{\Phi(D)} = \omega_{\Phi} \quad \text{is an } n\text{-form}$$

on R which can be expressed and ~~integrated~~ via the parameters.

ⓐ $n=1$: On a curve $\vec{c} : [a,b] \to \mathbb{R}^m$, with

$\omega = \sum_{i=1}^{m} f_i(\vec{x})\,dx_i = \vec{F} \cdot d\vec{s}$, we have

derivative of variables wrt parameter

$$\int_{\vec{c}} \omega = \int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int \omega_{\vec{c}}(\vec{c}\,'(t))\,dt$$

$$= \int_a^b \vec{F}(\vec{c}(t)) \cdot d\vec{s}(\vec{c}\,'(t))$$

$$= \int_a^b \vec{F}(\vec{c}(t)) \cdot \begin{bmatrix} dx_1(\vec{c}\,'(t)) \\ \vdots \\ dx_n(\vec{c}\,'(t)) \end{bmatrix} dt$$

$$= \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt$$

Note: This was our interpretation before, that

$$d\vec{s} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} = \begin{bmatrix} x_1'(t)dt \\ \vdots \\ x_n'(t)dt \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} dt = \vec{c}\,'(t)\,dt$$

(4b) $n=2$: On a surface $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^m$, with

$$\omega = \sum_{i,j=1}^{m} F_{ij}\, dx_i \wedge dx_j \quad \text{(neglecting simplifications)}$$

we have with $\Phi(D) = R$,

derivatives of variables wrt both parameters.

$$\int_R \omega = \int_{\Phi(D)} \omega_{\Phi}(\Phi_s, \Phi_t)\, ds \wedge dt, \text{ where}$$

Φ_s, Φ_t are the partial derivative vectors,

$$\Phi_s = \begin{bmatrix} \frac{\partial x_1}{\partial s} \\ \vdots \\ \frac{\partial x_m}{\partial s} \end{bmatrix}$$ of each $x_i = x_i(s,t)$ given

by the parameterization

$$\Phi(s,t) = (x_1(s,t), \cdots, x_n(s,t)).$$

Here $\int_R \omega = \int_{\Phi(D)} \omega_{\Phi}(\Phi_s, \Phi_t)\, ds \wedge dt = \int_R \vec{F} \cdot d\vec{S}$ where we

interpret $\vec{F}$ as a vector of F_{ij}'s (there will be $\binom{m}{2}$ of them after simplification), and

$d\vec{S}$ is a vector of corresponding $dx_i \wedge dx_j$

Both $\vec{F}$ and $d\vec{S}$ are $\binom{m}{2}$-vectors, after simplification.

(45) cont'd.

Then $\int_{\Phi} \omega_{\Phi}(\Phi_s, \Phi_t)\, ds \wedge dt = \int_D \vec{F}(\Phi(s,t)) \cdot d\vec{S}(\Phi_s, \Phi_t)$

where, for each $dx_i \wedge dx_j$ in $d\vec{S}$, we have

$$dx_i \wedge dx_j(\Phi_s, \Phi_t) = \begin{vmatrix} dx_i(\Phi_s) & dx_i(\Phi_t) \\ dx_j(\Phi_s) & dx_j(\Phi_t) \end{vmatrix} ds \wedge dt$$

$$= \begin{vmatrix} \frac{\partial x_i}{\partial s} & \frac{\partial x_i}{\partial t} \\ \frac{\partial x_j}{\partial s} & \frac{\partial x_j}{\partial t} \end{vmatrix} ds \wedge dt$$

In the special case where $n=2$ and $m=3$, so $\Phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $\omega = F_1(\vec{x})\, dy \wedge dz + F_2(\vec{x})\, dz \wedge dx + F_3(\vec{x})\, dx \wedge dy$.

Here $\binom{m}{2} = \binom{3}{2} = 3$

we set $\int_R \omega = \int_D \omega_{\Phi}(\Phi_s, \Phi_t)\, ds \wedge dt = \int \vec{F} \cdot d\vec{S}$, where

$\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$ is an actual vector field on $\mathbb{R}^3$ (has the right dim)

and $d\vec{S} = \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} \frac{\partial(y,z)}{\partial(s,t)} \\ \frac{\partial(z,x)}{\partial(s,t)} \\ \frac{\partial(x,y)}{\partial(s,t)} \end{bmatrix} ds \wedge dt$, as before with the vector surface integral.

Here is an example:

ex. Let $M = \{ (x,y,z) \in \mathbb{R}^3 \mid z = \sqrt{1-(x^2+y^2)} \}$

be the unit sphere above the xy-plane in $\mathbb{R}^3$,

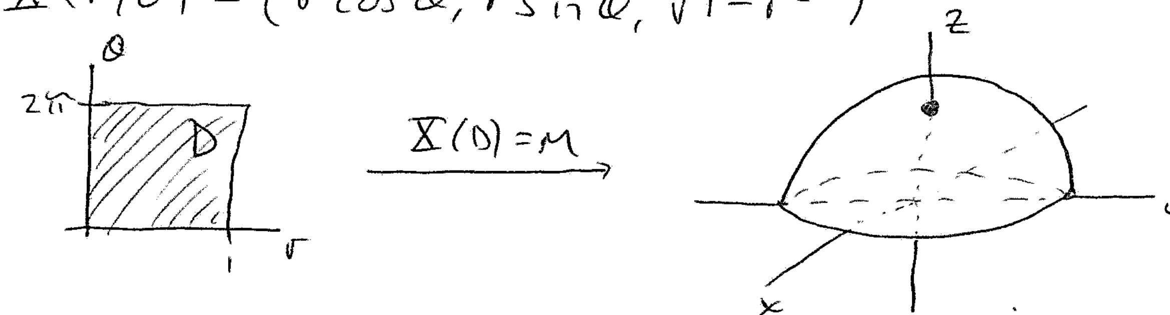
and $\omega = z^2 \, dx \wedge dy$ be a diff 2-form on $\mathbb{R}^3$.

Evaluate $\int_M \omega$.

Strategy: Parameterize the sphere and calculate the integral via the parameterization.

Solution: Parameterize the sphere as

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{1-r^2})$$



(each horizontal level set is a solid circle of radius $\sqrt{1-r^2}$).

Then $$\int_M \omega = \int_D \omega_{\Phi(r,\theta)} \left(\frac{\partial \Phi}{\partial r}(r,\theta), \frac{\partial \Phi}{\partial \theta}(r,\theta) \right) dr \wedge d\theta$$

$$= \int_D \omega_{\Phi(r,\theta)} \left(\begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{-r}{\sqrt{1-r^2}} \end{bmatrix}, \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \right) dr \wedge d\theta$$

Here, think of these as part of the cylindrical coordinate system on $\mathbb{R}^3$, with $z = 1 - r^2$.

$$= \int_D (1-r^2) \begin{vmatrix} dx \begin{bmatrix} \cos\theta \\ \sin\theta \\ -r/\sqrt{1-r^2} \end{bmatrix} & dx \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \\ dy \begin{bmatrix} \cos\theta \\ \sin\theta \\ -r/\sqrt{1-r^2} \end{bmatrix} & dy \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \end{vmatrix} dr \wedge d\theta$$

$$= \int_D (1-r^2) \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr \wedge d\theta$$

$$= \int_D (1-r^2) \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr \wedge d\theta$$

$$= \int_D (1-r^2)\, r\, dr \wedge d\theta = \int_0^{2\pi}\int_0^1 (r-r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 d\theta = \frac{\theta}{4}\Big|_0^{2\pi} = \frac{\pi}{2}.$$

---

exercise: Let $\vec{F} = \begin{bmatrix} y \\ x \end{bmatrix}$ be a vector field on $\mathbb{R}^2$.

Then for $d\vec{s} = \begin{bmatrix} dx \\ dy \end{bmatrix}$, the quantity $\omega = \vec{F} \cdot d\vec{s}$

is the 1-form $\omega = y\,dx + x\,dy$.

Calculate $\int_{\vec{c}} \omega$ where $\vec{c} : [0,2] \longrightarrow \mathbb{R}^2$, $\vec{c}(t) = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$.

(Hint: the answer is 32.)

ex. Integrating a 1-form on $\mathbb{R}$

Let $\omega = f(u)\,du$, a diff 1-form on $I = [c,d]$

Then $\int_I \omega = \int_c^d f(u)\,du$ like in Calculus I.

But lets reparameterize I via the function

$g: J \to I$, $g: \underbrace{[a,b]}_{x} \to \underbrace{[c,d]}_{u}$,

so that $u = g(x)$, $c = g(a)$, and $d = g(b)$.

Using the reparameterization (and the above example), we get

$$\int_I \omega = \int_J \omega_g(g'(x))\,dx = \int_a^b f(g(x)) \cdot g'(x)\,dx, \text{ so}$$

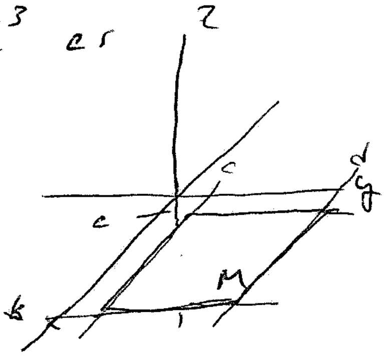
that
$$\int_{c=g(a)}^{d=g(b)} f(u)\,du = \int_a^b f(g(x)) \cdot g'(x)\,dx$$

Do you remember the Substitution Method in Calculus I??

ex. Why does Fubini's Thm hold when, in the language of forms, it looks like it shouldn't.

Let $D=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c\le y\le d\}$ be a rectangular region in $\mathbb{R}^2$. Embed D into $\mathbb{R}^3$ as

$$M=\{(x,y,0)\in\mathbb{R}^3 \mid (x,y)\in D\}$$



Let $\omega = f(x,y)\,dx\wedge dy$ be a diff. 2-form on D and extend to $\mathbb{R}^3$

$$\omega = 0\,dy\wedge dz + 0\,dz\wedge dx + f\,dx\wedge dy$$

Then $$\int_D \omega = \int_M \omega = \iint_M \vec{F}\cdot d\vec{S}\,,\quad \text{where } \vec{F}=\begin{bmatrix}0\\0\\f\end{bmatrix},\ d\vec{S}=\begin{bmatrix}dy\wedge dz\\dz\wedge dx\\dx\wedge dy\end{bmatrix}$$

$$=\iint_M f(x,y)\,dx\wedge dy$$

Note that it is absolutely the case that

$$\iint_M f(x,y)\,dx\wedge dy = \iint_M -f(x,y)\,dy\wedge dx.$$

Since this is a vector surface integral, orientation matters, and switching from $dx\wedge dy$ to $dy\wedge dx$ is a change of orientation in either $\mathbb{R}^2$ or $\mathbb{R}^3$.

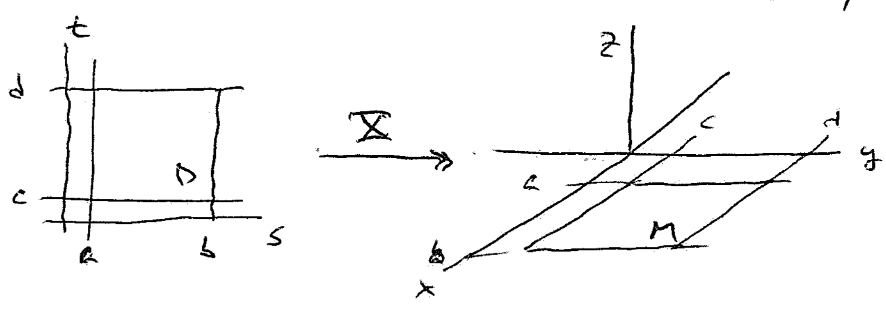
One way to "see" orientation directly is the following:

- If you parameterize a space, you automatically "induce" an orientation on the image.
- If you want the other orientation on the image you will have to introduce a minus sign.
- If you reparameterize with the other orientation, you will see the minus sign in your calculations.

Here, we parameterize M via D twice, using 2 different orientations. Since M can only have 1 orientation, we will see the minus sign.

---

① Let $\Phi : D \to M$, $\Phi(s,t) = (x,y,z)$,
$x(s,t) = s$
$y(s,t) = t$
$z(s,t) = 0$



Then $$\int_M \omega = \iint_D \omega_{\Phi}(\Phi_s, \Phi_t)\, ds \wedge dt = \iint_M \vec{F} \cdot d\vec{S}$$

And

$$\int_M \omega = \iint_D \omega_{\mathbf{X}}(\mathbf{X}_s, \mathbf{X}_t)\, ds \wedge dt$$

$$= \iint_D \vec{F}(\mathbf{X}(s,t)) \cdot (\mathbf{X}_s \times \mathbf{X}_t)\, ds\, dt, \quad \text{where again}$$

$$\vec{F}(\mathbf{X}(s,t)) = \begin{bmatrix} 0 \\ 0 \\ f(s,t) \end{bmatrix}, \quad \text{and}$$

$$d\vec{S} = (\mathbf{X}_s \times \mathbf{X}_t)\, ds\, dt = \begin{bmatrix} \frac{\partial(y,z)}{\partial(s,t)} \\ \frac{\partial(z,x)}{\partial(s,t)} \\ \frac{\partial(x,y)}{\partial(s,t)} \end{bmatrix} ds\, dt, \quad \text{so}$$

$$= \iint_D f(s,t) \cdot \frac{\partial(x,y)}{\partial(s,t)}\, ds\, dt, \quad \text{where}$$

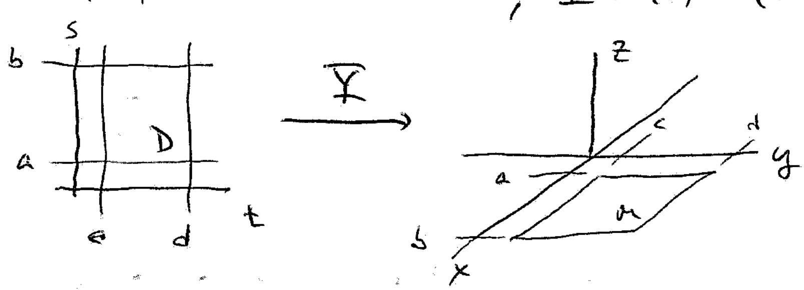
$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \text{so}$$

$$= \iint_D f(s,t)\, ds\, dt = \int_c^d \int_a^b f(s,t)\, ds\, dt.$$

---

(2) Now let $\mathbf{Y}: D \to M$, $\mathbf{Y}(t,s) = (x,y,0)$,

$x(t,s) = s$
$y(t,s) = t$
$z(t,s) = 0$



Here $$\int_M \omega = \iint_M \vec{F} \cdot d\vec{S} = \iint_D - \omega_{\mathbf{Y}}(\mathbf{Y}_t, \mathbf{Y}_s)\, dt \wedge ds$$

So $\int_M \omega = \iint_M \vec{F}\cdot d\vec{S} = \iint_D -\vec{F}(\Phi(t,s))\cdot(\Phi_t \times \Phi_s)\,dt\,ds$, since

$$d\vec{S} = (\Phi_t \times \Phi_s)\,dt\,ds = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial(x,y)}{\partial(t,s)} \end{bmatrix} dt\,ds = \begin{bmatrix} 0 \\ 0 \\ \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix} dt\,ds = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} dt\,ds$$

and

$$\int_M \omega = -\iint_D f(s,t)\cdot \frac{\partial(x,y)}{\partial(t,s)}\,dt\,ds$$

$$= -\iint_D f(s,t)\,(-1)\,dt\,ds = \int_c^d\int_a^b f(s,t)\,dt\,ds.$$

Hence there are 2 minus signs here that cancel out: one from the orientation mismatch between D and M and the other from the antisymmetry of the form.

Notes ① Of course, this generalizes to higher order forms in higher dimensional spaces, and Fubini still holds, over cuboidal regions.

② If the orientation on M doesn't matter, then the one given by the parameterization (the induced one) is the default.

Notation

For $\omega = \sum F_{i_1 i_2 \cdots i_n} \, dx_{i_1} \wedge \cdots \wedge dx_{i_n}$

a differential n-form in $\mathbb{R}^m$, $m \geq n$,

$$\int_M \omega = \underbrace{\int \cdots \int_M}_{n \text{ integrals}} \sum F_{i_1 \cdots i_n} \, dx_{i_1} \wedge \cdots \wedge dx_{i_n},$$

where M is a n-dimensional region in $\mathbb{R}^m$.

(The dimension of the form and the space must agree).

~~[illegible]~~

Def. ① Let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ be C^1. Then the exterior derivative of f, denoted df, is the 1-form

$$df = \frac{\partial f_1}{\partial x_1} dx_1 + \cdots + \frac{\partial f_n}{\partial x_n} dx_n = \nabla f \cdot d\vec{x}$$

② For $f: D \subset \mathbb{R}^n \to \mathbb{R}$, ~~[illegible]~~ C^1, we call f a 0-form.

So the exterior derivative of a 0-form is a 1-form.

Def (cont'd.)

③ Let $\omega = \sum F_{i_1 \cdots i_n} dx_{i_1} \wedge \cdots \wedge dx_{i_n}$ be a diff. n-form. Then

$$d\omega = \sum d(F_{i_1 \cdots i_n}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}$$

is the exterior derivative of ω and is a diff $(n+1)$-form.

Here, we wedge the differentiable 1-form $dF_{i_1 \cdots i_n}$ of each coord function $F_{i_1 \cdots i_n}$ and simplify (with lots of cancellations).

---

ex Let $\omega = x^2 y\, dx - x\, dy$ be a 1-form on $\mathbb{R}^2$.

Then
$$\begin{aligned} d\omega &= d(x^2 y) \wedge dx - d(x) \wedge dy \\ &= (2xy\, dx + x^2 dy) \wedge dx - dx \wedge dy \\ &= 2xy \overset{0}{\cancel{dx \wedge dx}} + x^2 dy \wedge dx - dx \wedge dy \\ &= -(1+x^2)\, dx \wedge dy. \end{aligned}$$

Q: What is $d(d\omega)$?

(Hint: Is it possible to have a 3-form on $\mathbb{R}^2$?)

ex: Let $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2 y e^{2z}$
Calculate df and $d(df) = d^2 f$.

Here, $df = 2xye^{2z}\,dx + x^2 e^{2z}\,dy + 2x^2 y e^{2z}\,dz$

And
$$
\begin{aligned}
d(df) &= d(2xye^{2z}) \wedge dx + d(x^2 e^{2z}) \wedge dy + (2x^2 y e^{2z}) \wedge dz \\
&= \cancel{(2y e^{2z}) dx \wedge dx} + \cancel{(2x e^{2z}) dy \wedge dx} + \cancel{(4xy e^{2z}) dz \wedge dx} \\
&\quad + \cancel{(2x e^{2z}) dx \wedge dy} + \cancel{0\, dy \wedge dy} + \cancel{(2x^2 e^{2z}) dz \wedge dy} \\
&\quad + \cancel{(4xy e^{2z}) dx \wedge dz} + \cancel{(2x^2 e^{2z}) dy \wedge dz} + \cancel{(4x^2 y e^{2z}) dz \wedge dz} \\
&= 0
\end{aligned}
$$

This is one of the features of <u>exterior differentiation</u>:

<u>Proposition</u> For ω a k-form, $d(d\omega) = d^2\omega = 0$.

<u>pf.</u> Mixed partials are equal for a C^1-function.

Other properties?

① If ω is a k-form, ν is a l-form, then

$$d(\omega \wedge \nu) = d\omega \wedge \nu + (-1)^{k+l} \omega \wedge d\nu$$

ⓐ exercise: Show $d(d(\omega \wedge \nu)) = 0$.

ⓑ If $k = l = 0$, then $f \wedge g = fg$, and this is just the product rule.

② Stick to forms on $\mathbb{R}^3$ (needs to be 3-dim).

d(0-form) = gradient

d(1-form) = curl of coefficient vector field

d(2-form) = divergence of coefficients vector field.

③ Only in $\mathbb{R}^3$, $\exists$ 1-1 correspondence between

$\binom{3}{0} = \binom{3}{3} = 1$ ⓐ 0-forms and 3 forms:

$$f \longleftrightarrow f\, dx \wedge dy \wedge dz$$

$\binom{3}{1} = \binom{3}{2} = 3$ ⓑ 1-forms and 2 forms

$$F_1\,dx + F_2\,dy + F_3\,dz \longleftrightarrow F_1\,dx \wedge dy + F_2\,dx \wedge dz + F_3\,dy \wedge dz.$$

Thm Let $D \subset \mathbb{R}^k$ be a compact region w/ nonempty interior, and $M = \Phi(D)$ be an oriented, parameterized k-dimensional hypersurface in $\mathbb{R}^n$, $n \geq k$, with ∂M if any, oriented compatibly.

For ω a $(k-1)$-form defined on an open set in $\mathbb{R}^n$ containing M, we have

$$\int_M d\omega = \int_{\partial M} \omega$$

Notes ① Called Generalized Stoke's Thm.

② If $\partial M = \emptyset$, then $\int_{\partial M} \omega = 0$.

(integrating over the empty set).

③ Let $k=n=3$. The thm of Gauss.

Here, $D=M$ a bounded closed region in $\mathbb{R}^3$,

and $\omega = F_1(\bar{x})\,dy\wedge dz + F_2(\bar{x})\,dz\wedge dx + F_3(\bar{x})\,dx\wedge dy$

Parameterize ∂M via $\vec{X}: R\subset\mathbb{R}^2 \to \mathbb{R}^3$ w/ $\vec{X}(s,t)=\partial M$

Then $\displaystyle\int_{\partial M}\omega = \iint_R \vec{F}(\vec{X}(s,t))\cdot d\vec{S}$, for $\vec{F}=\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix}$, $d\vec{S}=\begin{bmatrix}dy\wedge dz\\dz\wedge dx\\dx\wedge dy\end{bmatrix}$

$$= \iint_R \vec{F}(\vec{X}(s,t))\cdot\vec{N}(s,t)\,dS$$

where $d\vec{S}=\vec{N}(s,t)\,dS = \begin{bmatrix}\frac{\partial(y,z)}{\partial(s,t)}\\ \frac{\partial(x,z)}{\partial(s,t)}\\ \frac{\partial(x,y)}{\partial(s,t)}\end{bmatrix}dS$

$$= \iint_R F_1(\vec{x}(s,t))\,\frac{\partial(y,z)}{\partial(s,t)}\,ds\,dt + \cdots = \boxed{\text{RHS (Gauss)}}$$

And $\displaystyle\int_M d\omega = \int_M d\left(F_1\,dy\wedge dz + F_2\,dz\wedge dx + F_3\,dx\wedge dy\right)$

$$= \int_M \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right)\wedge dy\wedge dz$$

$$+ \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right)\wedge dz\wedge dx$$

$$+ \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right)\wedge dx\wedge dy$$

$$= \int_M\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\wedge dy\wedge dz = \int_M (\operatorname{div}\vec{F})\,dV$$

$= \text{RHS (Gauss)}$

(4) Let $k=2$, $n=3$. The Thm of Stoke's.

Here $D \subset \mathbb{R}^2$ is a compact region, with $M = \Phi(D) \subset \mathbb{R}^3$ an oriented parametrized surface in 3-space, ∂M a finite set of closed curves oriented compatibly.

For $\omega = F_1\,dx + F_2\,dy + F_3\,dz$ a 1-form in $\mathbb{R}^3$,

Then $\int_{\partial M} \omega = \int_{\vec{c}} \vec{F}(\vec{c}(t)) \cdot d\vec{s}$, where $\vec{c}: I \to \partial M$ and $d\vec{s} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$.

$$= \int_I \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt = \boxed{RHS\text{(Stoke's)}}$$

And
$$\int_M d\omega = \int_M d(F_1\,dx + F_2\,dy + F_3\,dz)$$
$$= \int_M \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right)\wedge dx$$
$$+ \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right)\wedge dy$$
$$+ \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right)\wedge dz$$
$$= \int_M \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)dy\wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)dz\wedge dx$$
$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial z}\right)dx\wedge dy$$
$$= \int_M \nabla\times\vec{F}\cdot d\vec{S} = \boxed{LHS\text{ (Stoke's)}}$$

(5) Let $k = n = 2$. The thm of Green.

Here, $D \subset \mathbb{R}^2$ is a compact region w/ boundary ∂D in $\mathbb{R}^2$, and $\omega = \sum_{i=1}^{2} F_i \, dx_i = \vec{F} \cdot d\vec{s}$

Parameterize ∂D as $\vec{c} : [a,b] \longrightarrow \mathbb{R}^2$

Then
$$\int_{\partial D} \omega = \int_{\vec{c}} \vec{F}(\vec{c}(t)) \cdot d\vec{s} = \int_I \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$
$$= \int_I F_1 \, dx + F_2 \, dy = \boxed{\text{RHS (Green}}$$

And
$$\int_D d\omega = \int_D d\left(\sum_{i=1}^{2} F_i \, dx_i \right) = \int_D \sum_{i=1}^{2} dF_i \wedge dx_i)$$
$$= \int_D \left(\frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy \right) \wedge dx + \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy \right) \wedge dy$$
$$= \int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx \wedge dy = \text{LHS (Green)}.$$

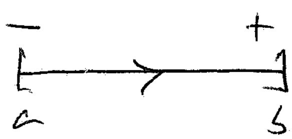
(6) Let $k=n=1$.

Here, ω = a continuous fnc = 0-form
and $D = M$ = interval $[a,b] = I$.
so $\partial D = \{a,b\}$ = 2 pts, $\omega = f|_I$.

$\int_{\partial D}\omega$ = adding up all values of $f(x)$ on the set of pts $\{a,b\}$, oriented compatibly with I:

$= f(b) - f(a)$

(Here, if A is a discrete set of pts, $\int_A f = \Sigma f(a)$.
but each pt must be oriented, and with



, $f(a)$ is negative.)

And $\int_D d\omega = \int_I df = \int_a^b f'(x)\,dx$.

Here $$\int_{\partial D}\omega = \underbrace{f(b)-f(a) = \int_a^b f'(x)\,dx}_{\text{Fundamental Thm of Calculus.}} = \int_D d\omega$$

Generalized Stokes's is the Fundamental Thm of Calculus in vector Calculus (dimensionless).