CONCEPT: SECTION 4.4: L'HOSPITAL'S RULE

110.109 CALCULUS I (PHYS SCI & ENG)
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Here is a special case proof of the theorem known as L'Hospital's Rule. Recall that this rule establishes that under certain circumstances, the limit of a ration of functions is equal to the limit of the ration of the derivatives of the functions.

Theorem 1 (L'Hospital's Rule)**.** *Suppose $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ on an open interval containing a point a (except possibly at a). Suppose also that the Quotient Limit Law, if applied directly to the limit of the quotient, yields the indeterminate forms $\frac{0}{0}$, or $\pm\frac{\infty}{\infty}$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provide that the limit on the right-hand-side either exists or is $\pm\infty$.

The proof of the general theorem can be found in Appendix F. But for now, a very brief proof can be given in the specific case that $f(a) = g(a) = 0$, that $f'(x)$ and $g'(x)$ are continuous at $x = a$, and that $g'(a) \neq 0$.

Proof. So let us suppose that $f(a) = g(a) = 0$, that $f'(x)$ and $g'(x)$ are continuous at $x = a$, and that $g'(a) \neq 0$. Then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

by continuity. But by the definition of the derivative, we have

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}.$$

The Quotient Limit Law will apply here since both limits exist (and the denominator is not 0), and

$$\frac{\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}.$$

Now since everything is under one limit, you can see what cancels. We get

$$\lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

since we assumed that $f(a) = g(a) = 0$. But that is the theorem in this special case. □

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