HOMEWORK SET 6. SELECTED SOLUTIONS

AS.110.106 CALCULUS I (BIO & SOC SCI)
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1. Selected Exercises

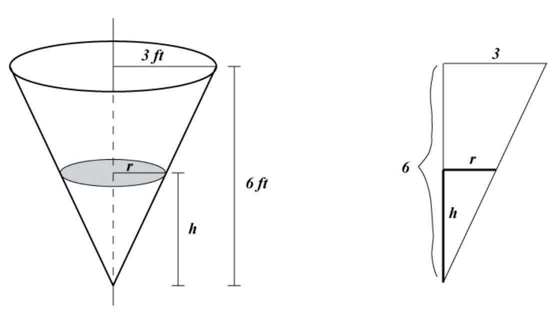
Exercise (4.4.38). Assume that $f(x)$ and $g(x)$ are differentiable. Find $\dfrac{d}{dx} f\left[\dfrac{1}{g(x)}\right]$.

Strategy. Considering $\frac{1}{g(x)}$ as simply another function, differentiate $f\left[\frac{1}{g(x)}\right]$ by using the Chain Rule. When it is time to differentiate $\frac{1}{g(x)}$, use the Quotient Rule.

Solution. Identify the outer function in the expression $f\left[\frac{1}{g(x)}\right]$ as $f(x)$ and the inner function of the composition as $\frac{1}{g(x)}$. Then

$$\begin{aligned}
\frac{d}{dx} f\left[\frac{1}{g(x)}\right] &= f'\left[\frac{1}{g(x)}\right] \cdot \frac{d}{dx}\left[\frac{1}{g(x)}\right] && \text{Chain Rule} \\
&= f'\left[\frac{1}{g(x)}\right] \cdot \frac{0 \cdot g(x) - 1 \cdot g'(x)}{(g(x))^2} && \text{Quotient Rule} \\
&= -f'\left[\frac{1}{g(x)}\right] \cdot \frac{g'(x)}{(g(x))^2}.
\end{aligned}$$

Exercise (4.4.70). Suppose that we pump water into an inverted right circular conical tank at the rate of 5 cubic feet per minute (i.e., the tank stands with its point facing downward). The tank has a height of 6 ft and the radius on top is 3 ft. What is the rate at which the water level is rising when the water is 2 feet deep? (Note that the volume of a right circular cone of radius r and height h is $V = \dfrac{1}{3}\pi r^2 h$.)



Strategy. Drawing the tank, we see that we can solve this problem by using the volume formula to relate the rates of change with respect to time, of the three quantities V, r and h. Here, the rate of volume change is constant, and we can relate the changing radius r and height h of the cone via similar triangles by slicing through the cone and using the full height and radius.

Solution. Notice that the formula for volume relates both radius and height to volume, so that if volume is changing with respect to time, then so it the height h and the radius r of the cone which contains the water. We have

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{1}{3}\pi r^2 h\right] = \frac{1}{3}\pi\left(\frac{d}{dt}\left[r^2\right]\cdot h + r^2\cdot\frac{dh}{dt}\right) = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right),$$

using the Product Rule and the Chain Rule.

Now the quantities we do know are $\frac{dV}{dt} = 5$ cubic feet per minute, and at the moment we are considering, we have $h = 2$ feet and are looking for $\frac{dh}{dt}$. To solve this problem, we need to also know r and $\frac{dr}{dt}$ at this moment. However, looking at the similar triangles on the right side of the figure, we immediately see that for any height h, we have the ratio

$$\frac{r}{h} = \frac{3}{6}.$$

Thus $r = \frac{3}{6}h = \frac{h}{2}$, and when $h = 2$, we have $r = 1$.

Also, we can use the same relationship between r and h to relate the rates of change:

$$r = \frac{h}{2}, \quad \Longrightarrow \quad \frac{dr}{dt} = \frac{d}{dt}\left[\frac{h}{2}\right] = \frac{1}{2}\frac{dh}{dt}.$$

Using both of these facts together, we can rewrite the related rates equation as

$$\begin{aligned}\frac{dV}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) &= \frac{1}{3}\pi\left(2\left(\frac{h}{2}\right)h\left(\frac{1}{2}\frac{dh}{dt}\right) + \left(\frac{h}{2}\right)^2\frac{dh}{dt}\right)\\ &= \frac{1}{3}\pi\left(\frac{3}{4}h^2\frac{dh}{dt}\right)\\ &= \frac{\pi}{4}h^2\frac{dh}{dt}.\end{aligned}$$

Thus, since $\frac{dV}{dt} = 5$ always, then at $h = 2$, we obtain

$$5 = \frac{\pi}{4}(2)^2\frac{dh}{dt}, \quad \Longrightarrow \quad \left.\frac{dh}{dt}\right|_{h=2} = \frac{5}{\pi} \text{ feet per minute.}$$