110.106 CALCULUS I: BIOLOGICAL AND SOCIAL SCIENCES

A Limit Example: $g(x) = \dfrac{\sin x}{x}$

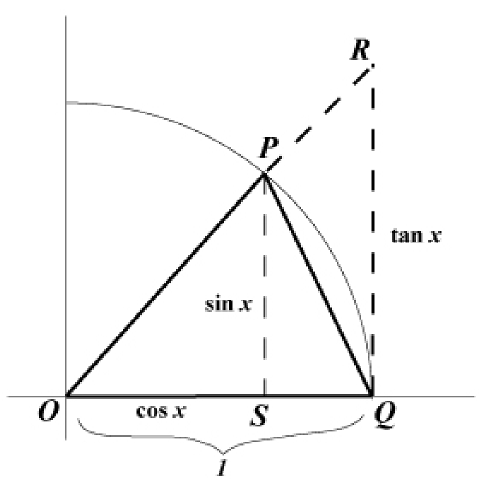
The book has a good calculation using the Squeezing Theorem to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Here, I will write a slightly different one, and take a bit more time with the details. I hope that following this will give you some additional insight on how limits work and can be calculated in some interesting ways:

Claim. $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1.$

Proof. We will show explicitly that $\displaystyle\lim_{x \to 0^+} \frac{\sin x}{x} = 1$, like in the book. The argument for the other side limit is almost exactly the same. Let $g(x) = \dfrac{\sin x}{x}$. Consider the drawing:



For a circle of radius 1 centered at the origin, we know the following:

- the coordinates of the point P are $P = (\cos x, \sin x)$.
- The area of the sector $\sphericalangle(OPQ) = \pi \cdot (1)^2 \cdot \frac{x}{2\pi} = \frac{x}{2}$. (See inside back cover of the book for this.)

- The area of the triangle $\triangle(OPQ) = \frac{1}{2}(1)(\sin x)$ since the base is the radius of the circle and the height is the vertical coordinate of P.
- The area of the triangle $\triangle(ORQ) = \frac{1}{2}(1)(\tan x)$. Here again, the base is the radius of the circle. The height, though, is found via the fact that the two right triangles $\triangle(OPS)$ and $\triangle(ORQ)$ are similar. Hence the ratio of their respective right angle sides are equal. The base of $\triangle(OPS)$ is $\cos x$ (why?), so $\frac{\text{height}}{\text{base}} = \frac{\sin x}{\cos x} = \frac{\text{height}(\triangle(ORQ))}{1}$ and, of course, $\frac{\sin x}{\cos x} = \tan x$.

It should be obvious to see the following relationship:

$$\text{area of } \triangle(OPQ) < \text{area of } \sphericalangle(OPQ) < \text{area of } \triangle(ORQ),$$

so that

$$\frac{1}{2}\sin x < \frac{x}{2} < \frac{1}{2}\tan x.$$

But this means

$$\sin x < x < \tan x,$$

at least when $x \in \left(0, \frac{\pi}{2}\right)$. Also, for $x \in \left(0, \frac{\pi}{2}\right)$, neither x nor $\sin x$ nor $\tan x$ are 0, and all are positive. Hence inverting this inequality is possible and

$$\frac{1}{\sin x} < \frac{1}{x} < \frac{1}{\tan x} = \frac{\cos x}{\sin x}.$$

Now multiply the entire inequality by $\sin x$ to get

$$1 < \frac{\sin x}{x} < \cos x.$$

Call $f(x) = 1$ and $h(x) = \cos x$. Note that f and h are both continuous at $x = 0$ and $g(x)$ is continuous near $x = 0$ (but not at $x = 0$!). And not also that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} h(x) = 1.$$

Hence by the Squeezing Theorem, it now follows that

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{\sin x}{x} = 1.$$

Hope this helps. □