Class 5: Section 2.2

Given the model of the previous class,

$$N(t) = N_0 a^t, \quad t \in \mathbb{N}$$

we created a list of values N_0, $N_1 = N(1)$, $N_2 = N(2)$, N_3, ...

An infinite list of real numbers is called a <u>sequence</u>, and denoted with brackets.

ex. $\{1, 2, 4, 6, 10, e, 2.6, \ldots\}$.

or via a variable, using a subscript to denote position in the list:

$$\{a_n\} = \{a_0, a_1, a_2, \cdots\}$$

Sometimes the list starts at a position other than the 0th position:

ex. $\{a_n\}$, where $a_n = \frac{1}{n}$. (a_0 doesn't make sense here).

In this last example, the nth term a_n is given by an expression (a function) so that it can be calculated ($a_{10} = \frac{1}{10}$, $a_{47} = \frac{1}{47}$, ...)

Sometimes we denote explicitly what the first position is:

ex. $\{b_n\}_{n=2}^{\infty}$, or $\{\frac{1}{n}\}_{n=1}^{\infty}$

Sometimes we just give the expression:

ex. $a_n = 2^n + 1$

Here we can calculate $\{a_n\} = \{1, 3, 5, 9, 17, \ldots\}$.

ex $b_n = (-1)^n \frac{1}{n+1}$, so $\{b_n\} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5} \ldots\}$.

ex. $c_n = f(n)$, $f: \mathbb{N} \to \mathbb{R}$, $f(n) = \sin(\frac{\pi}{2} n)$

Here $\{c_n\} = \{0, 1, 0, -1, 0, 1, 0, -1, \ldots\}$

ex. $N_i = N_0 a^t$, $a > 0$, $a \neq 1$. Recognize this?

Sometimes, a sequence is defined only <u>recursively</u> by a function or expression:

ex. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax$, $a > 0$, $a \neq 1$.

Then we can use f to create a sequence by specifying ① a starting term (needed) and ② a rule for calculating the next terms.

ex. Let $\{N_i\}$ be the sequence defined by

$$N_0 = 100, \quad N_{i+1} = f(N_i) \text{ for all } i \in \mathbb{N}.$$

where $f(x) = 7x$ (let $a=7$)

Then

$$N_0 = 100$$
$$N_1 = f(N_0) = f(100) = 7(100) = 700$$
$$N_2 = f(N_1) = f(700) = 7(700) = 4900$$
$$\vdots$$

Then $\{N_i\} = \{100, 700, 4900, 34300, \ldots\}$.

$= \{100, 100 \cdot 7, 100 \cdot 7^2, 100 \cdot 7^3, \ldots\}$

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ex. Write the first few terms of the sequence $\{b_n\}$, where $b_{n+1} = \frac{1}{4} b_n + \frac{3}{4}$, $b_0 = 2$.

There are not a lot of special properties associated to a sequence. There is one, tho.

Q: How does a sequence behave in the long run? As the position goes to infinity?

Q: Does the sequence tend toward a single real number?

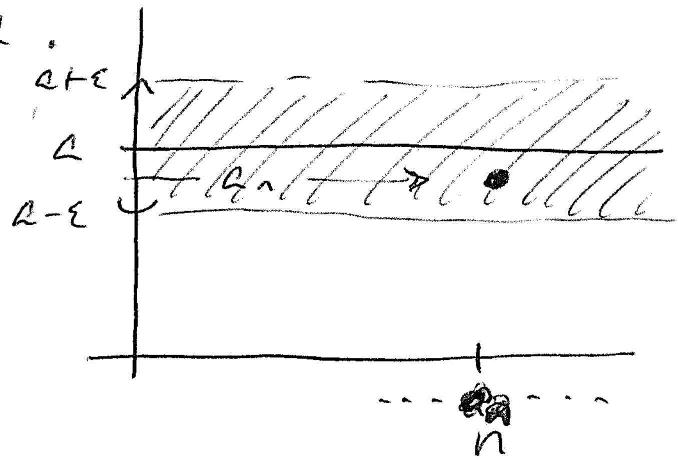
<u>Def</u>. A sequence $\{a_n\}$ has a limit a, written $\lim_{n\to\infty} a_n = a$, or $\{a_n\} \to a$, if for every (small #) $\varepsilon > 0$, there is a natural number N so that

$$|a_n - a| < \varepsilon \quad \text{whenever} \quad n > N$$

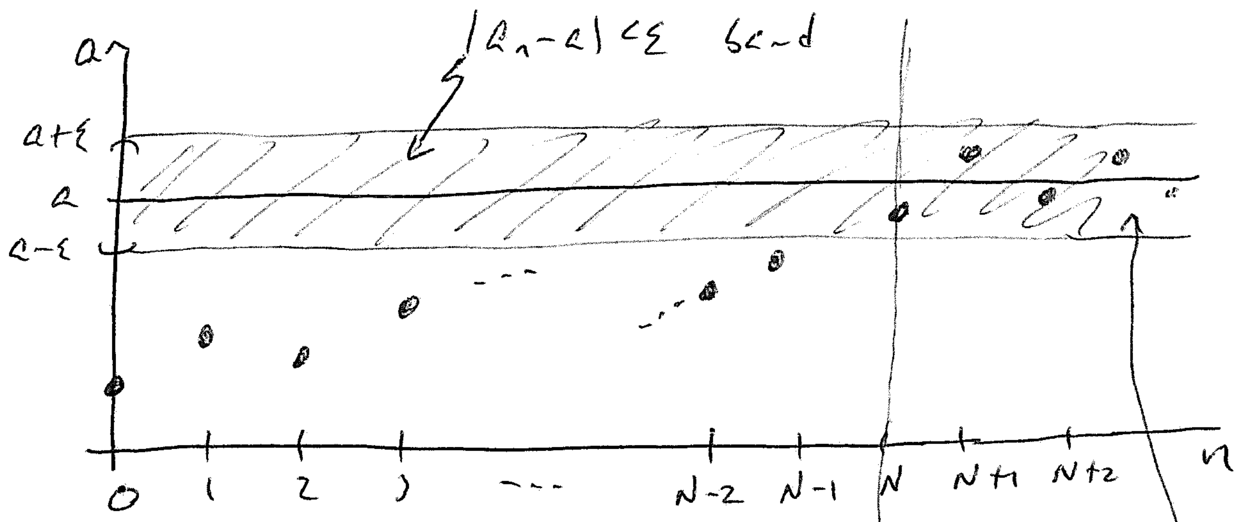
<u>Notes</u>: ① If limit exists, we say $\{a_n\}$ <u>converges</u> to a or $\{a_n\}$ is <u>convergent</u>. Else $\{a_n\}$ is divergent.

(2) The phrase "$|a_n - a| < \varepsilon$" means that the term a_n is less than ε-distance away from a.

$|a_n - a| < \varepsilon$
is the same as
$a - \varepsilon < a_n < a + \varepsilon$



(3) Visually (geometrically),



Given an $\varepsilon > 0$, the N is chosen to be the index value so that for ever more after N, (for all $n > N$) the sequence values a_n all stay in the band defined by ε.

(4) Typically, the smaller the $\varepsilon > 0$, the thinner this band, and the larger the value of N (the farther out along the sequence) will be.

(5) Sequence limits play a similar role as horizontal asymptotes do, if you have heard of such a thing.

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ex. Does the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converge?

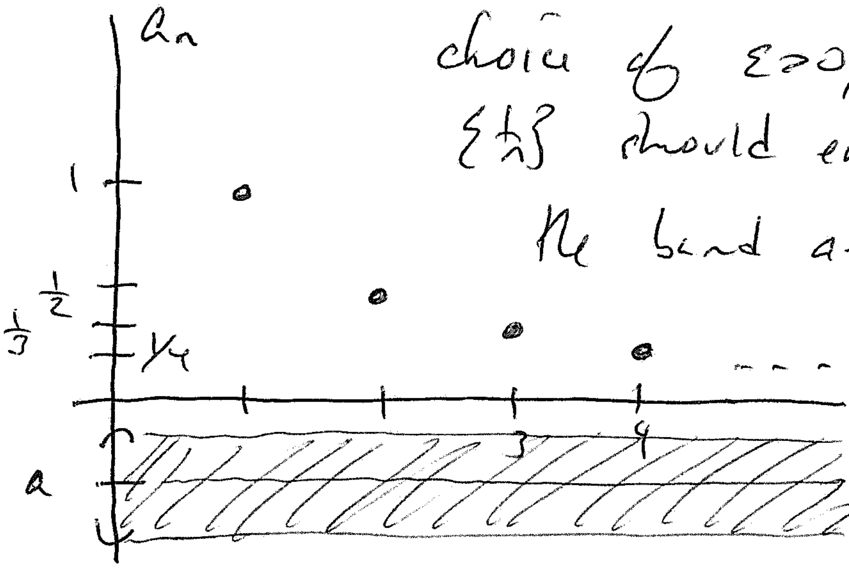
Strategy: Use the definition to construct an N for any $\varepsilon > 0$ chosen.

First, though, an analysis of possibilities:

Q1: Is it possible that $\lim_{n\to\infty} \frac{1}{n} < 0$?

A1: The sequence is always positive and decreasing.

Suppose $\lim_{n\to\infty} \frac{1}{n} = a < 0$. Then for any choice of $\varepsilon > 0$, the sequence $\{\frac{1}{n}\}$ should eventually enter the band $a - \varepsilon < a_n = \frac{1}{n} < a + \varepsilon$



Choose $\varepsilon > 0$ small enough so that entire band $(a-\varepsilon, a+\varepsilon)$ is below x-axis.

Choose $0 < \varepsilon < |a|$. As long as $|a| > 0$, choose $\varepsilon = \frac{|a|}{2}$ for example. Then $\varepsilon > 0$, and this works. But then $\{\frac{1}{n}\}$ never enter the shaded band above.

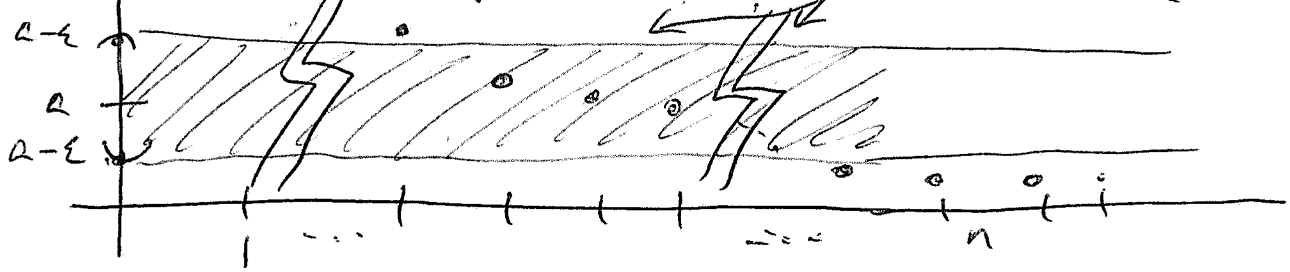
A1: No.

Q2: Is it possible that $\lim_{n\to\infty} \frac{1}{n} = a > 0$?

A2: Suppose this is possible: $\lim_{n \to 0} \frac{1}{n} = a > 0$.

Then again, choose $\varepsilon = \frac{a}{2} > 0$. Entire band $(a-\varepsilon, a+\varepsilon)$ lies above x-axis.

If $a > 0$ is small, then at some pt $\{\frac{1}{n}\}$ enters band. But the

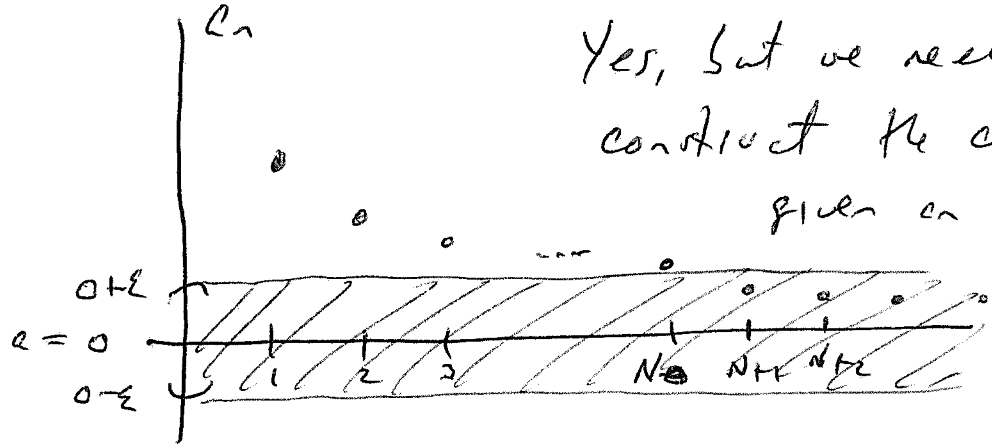


bottom of the band is still positive. At some later pt the sequence $\{\frac{1}{n}\}$ will leave the band and never return. This will happen when subscript n gets larger than $\frac{1}{a-\varepsilon}$ (don't worry about this part though)

A2: No.

Q3: Is it possible that $\lim_{n\to\infty} \frac{1}{n} = 0$?

A3: It does look like for any $\varepsilon > 0$, we can create an interval band around $a = 0$ and at some point the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ will enter it and never leave.



Yes, but we need to actually construct the choice of N given an $\varepsilon > 0$ for the limit to actually be 0.

For example, suppose we choose $\varepsilon = .1 = \frac{1}{10}$.

Then the condition of the definition $|a_n - a| < \varepsilon$ becomes $|\frac{1}{n} - 0| < \frac{1}{10}$, or $\frac{1}{n} < \frac{1}{10}$.

Can you find a position in the sequence $\{\frac{1}{n}\}$ where after that position N the sequence lives in the band $(-\varepsilon, \varepsilon)$ forever more?

Yes, choose $N = 10$. Then for $n > 10$, $\frac{1}{n} < \frac{1}{10}$.
(why does this work?)

If we chose $\varepsilon = \frac{1}{100}$, the condition becomes $\frac{1}{n} < \varepsilon = \frac{1}{100}$. Choose $N = 100$.

If $\varepsilon = \frac{3}{50}$? The condition becomes $\frac{1}{n} < \frac{3}{50}$. Play with this to get $\frac{50}{3} < n$.

Then $\frac{1}{n} < \frac{3}{50}$ when $n > \frac{50}{3} = 16.66\overline{6}$. So choose $N = 17$. Then $\frac{1}{n} < \frac{3}{50}$ when $n > 17$.

In general $\frac{1}{n} < \varepsilon$ precisely when $\frac{1}{\varepsilon} < n$.

Hence choose $N = \frac{1}{\varepsilon}$ (or the next integer).

Then when $n > N = \frac{1}{\varepsilon}$, we have $\frac{1}{n} < \varepsilon$.

We have ~~chosen~~ a way to choose N given <u>any</u> $\varepsilon > 0$ so that when $n > N$, we know $|a_n - a| = |\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon$.

Hence $\lim_{n \to \infty} \frac{1}{n} = 0$. ∎

Follow up questions:

We can modify this construction to show that

(i) $\lim_{n\to\infty} \frac{1}{n+1} = 0$

(ii) $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

(iii) $\lim_{n\to\infty} \frac{(-1)^n}{n+1} = 0.$