Class 6: Sectn 2.2.

Pay close attention to the limit laws on page 74. They will be the primary means for you to "calculate" limits of new functions using limits of sequences you already know.

But in particular, pay attention to one crucial fact: Certain limits must exist for the laws to apply!

example: The Sum Law of limits

If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist,

then $\lim_{n\to\infty} (a_n+b_n) = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} b_n\right)$

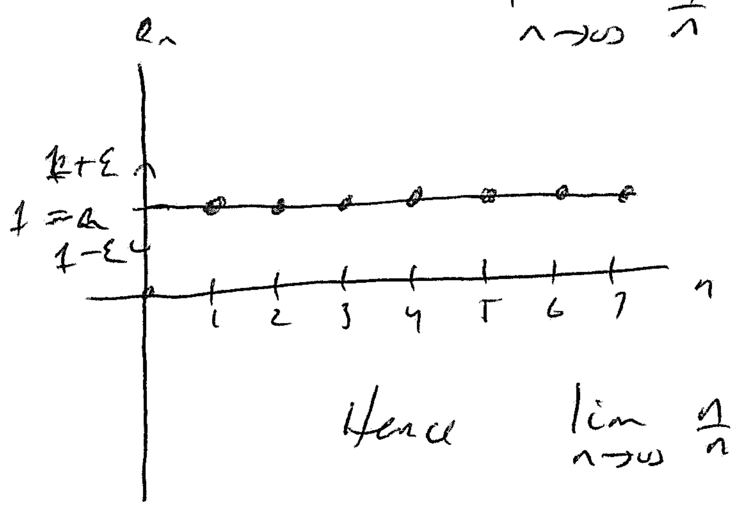
Caution: The law cannot be used if the 2 limits on the Right hand side do not both exist!!

<u>example</u> Let $a_n = \frac{n}{n}$. Does $\{\frac{n}{n}\}_{n=1}^{\infty}$ have a limit? For $n > 0$, $\frac{n}{n} = 1$ for all $n \in \mathbb{N}$.

Hence $\{\frac{n}{n}\} = \{1\} = \{1, 1, 1, \ldots\}$.

By the Definition of a limit for sequences, choose $a = 1$, and check to see $\lim_{n\to\infty} \frac{n}{n} = 1$. Given any $\varepsilon > 0$, if I choose $N = 1$, then if $n > N = 1$, I know $|a_n - a| < \varepsilon$

$$0 = \left|\frac{n}{n} - 1\right| < \varepsilon \quad \checkmark$$



Hence $\lim_{n\to\infty} \frac{n}{n} = 1$.

<u>The Product Rule for limits</u>: if $\lim_{n\to\infty} a_n$ and ~~limit~~ $\lim_{n\to\infty} b_n$ both exist, then

$$\lim_{n\to\infty}(a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right)$$

Try it: $\lim_{n\to\infty} \frac{n}{n} = \lim_{n\to\infty}\left(n \cdot \frac{1}{n}\right) \overset{?}{=} \left(\lim_{n\to\infty} n\right) \cdot \left(\lim_{n\to\infty} \frac{1}{n}\right)$

$$\underset{1}{\shortparallel} \qquad\qquad \underset{\infty}{\shortparallel} \cdot \underset{0}{\shortparallel} = ?$$

But since the 2 individual limits on the RHS do not both exist, the rule does not apply.

<u>example</u> Find $\lim_{n\to\infty} \frac{4n+1}{2n}$, if ~~it~~ exists.

<u>Strategy</u>: Use limits Laws as they apply to find limits based on known limits.

<u>Solution</u>: $\lim_{n\to\infty} \frac{4n+1}{2n} = \lim_{n\to\infty}\left(\frac{4n}{2n} + \frac{1}{2n}\right)$

$$\overset{\text{only if both}}{\underset{\text{exist}}{=}} \left(\lim_{n\to\infty} \frac{4n}{2n}\right) + \left(\lim_{n\to\infty} \frac{1}{2n}\right)$$

(Sum Law)

constant multiple rule.

$$= \left(\lim_{n\to\infty} 2\right) + \frac{1}{2}\cdot \lim_{n\to\infty} \frac{1}{n}$$

$$= 2 + \frac{1}{2}(0) = 2 + 0 = \boxed{2}.$$

Hence $\boxed{\lim_{n\to\infty} \frac{4n+1}{2n} = 2}$

Note: In the previous example, the 2 individual limits <u>did</u> exist. Hence the Sum Rule was valid.

<u>example</u> $\lim_{n\to\infty} \frac{4n+1}{2n} \neq \frac{\lim_{n\to\infty} 4n+1}{\lim_{n\to\infty} 2n}$

Q: Why does the Quotient Rule for limits not apply here?

exaple Find $\lim_{n\to\infty} \frac{1}{n^2}$! $\lim_{n\to\infty} \frac{(-1)^n}{n^2}$ IV

example Explain why $\lim_{n\to\infty} \sin\left(\frac{n\pi}{2}\right)$

does not exist. but $\lim_{n\to\infty} \sin n\pi$ does.

Back to recursions

Q: What is the difference between the 2 sequences types?

Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = rx$, $r \neq 0$

(a) $a_n = f(n)$

(b) $b_{n+1} = f(b_n)$

(1) Write out the 1st few terms:

$\{a_n\} = \{0, r, 2r, 3r, 4r, 5r, \cdots\}$.

For b_n, we cannot start without a starting value b_0:

Let $b_0 = 0$, then $\{b_n\} = \{0, 0, 0, 0, \cdots\}$.

Let $b_0 = 1$, then $\{b_n\} = \{1, r, r^2, r^3, r^4, r^5, \cdots\}$.

Another example

If it exists, find $\lim_{n\to\infty} c_n$, where $c_n = \dfrac{2n^4-6}{1-n^3-3n^4}$

Here:

- Algebraic manipulations are useful
- Cannot use the Product Rule or Quotient Rule directly.
- Consider a clever form of 1.

Note that $c_n = \dfrac{2n^4-6}{1-n^3-3n^4}\left(\dfrac{\frac{1}{n^4}}{\frac{1}{n^4}}\right) = \dfrac{2-\frac{6}{n^4}}{\frac{1}{n^4}-\frac{1}{n}-3}$ why did we choose $\frac{1}{n^4}$ here?

Now the Quotient Rule here is possible:

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{2-\frac{6}{n^4}}{\frac{1}{n^4}-\frac{1}{n}-3} \overset{?}{\underset{QR}{=}} \frac{\lim_{n\to\infty} 2-\frac{6}{n^4}}{\lim_{n\to\infty} \frac{1}{n^4}-\frac{1}{n}-3}$$

$$\overset{?}{\underset{\text{Sum Rule}}{=}} \frac{\overbrace{\lim_{n\to\infty} 2}^{2} - 6\overbrace{\lim_{n\to\infty}\frac{1}{n^4}}^{0}}{\underbrace{\lim_{n\to\infty}\frac{1}{n^4}}_{0} - \underbrace{\lim_{n\to\infty}\frac{1}{n}}_{0} - \underbrace{\lim_{n\to\infty} 3}_{-3}} = \frac{2}{-3} = -\frac{2}{3}$$

As the individual limits all exist, the Sum Rule and hence the Quotient Rule is valid and hence we can use them.

② For different values of r, do these sequences have limits?

Let $r=3$: $\{a_n\} = \{0, 3, 6, 9, 12, 18, \cdots, 3n, \cdots\}$.

$b_0=0$ $\{b_n\} = \{0, 0, 0, 0, 0, \cdots\}$.

$b_0=1$ $\{b_n\} = \{1, 3, 9, 27, 81, \cdots, 3^n, \cdots\}$.

Let $r=\frac{1}{3}$: $\{a_n\} = \{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, \cdots, \frac{n}{3}, \cdots\}$.

$b=0$ $\{b_n\} = \{0, 0, 0, \cdots\}$.

$b=1$ $\{b_n\} = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \cdots, \frac{1}{3^n} = (\frac{1}{3})^n, \cdots\}$.

Let $r=-2$: $\{a_n\} = \{0, -2, -4, -6, -8, \cdots, (-2)n, \cdots\}$.

$b=0$ $\{b_n\} =$ well, you know ...

$b=1$ $\{b_n\} = \{1, -2, 4, -8, 16, -32, 64, \cdots, (-2)^n, \cdots\}$

Let $r=-\frac{1}{2}$: $\{a_n\} = 0, -\frac{1}{2}, -1, -\frac{3}{2}, \cdots, -\frac{n}{2}, \cdots\}$.

$b=1$: $\{b_n\} = \{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \cdots, \frac{1}{(-2)^n}, \cdots\}$.

③ How many of these sequences have limits?
At least for the positive values of r, if these sequences model populations, how many have fixed populations?

__example__ Another recursively defined sequence.

Suppose $a_{n+1} = \frac{1}{4} a_n + \frac{3}{4}$ models a population of ants (a_n measured in millions), where $a_0 > 0$ (doesn't make much sense when $a_0 = 0$; why not?).

Choose __ANY__ starting value (population) $a_0 > 0$.

Does $\lim\limits_{n \to \infty} a_n$ exist? If so, find it.

(a) Choose $a_0 = 2$. Then $\{a_n\} = \{2, \frac{5}{4}, \frac{17}{16}, \frac{65}{64}, \frac{257}{256}, \ldots\}$.

(b) Choose $a_0 = \frac{1}{2}$. Then $\{a_n\} = \{\frac{1}{2}, \frac{7}{8}, \frac{31}{32}, \frac{127}{128}, \ldots\}$.

(c) Choose $a_0 = 10$. Then $\{a_n\} = \{10, \frac{13}{4}, \frac{25}{16}, \frac{73}{64}, \ldots\}$.

Can you guess the limits if they exist in each case?

To calculate like we have been doing, we would need the find the nth term (as a function of n). This is not easy and sometimes impossible!

Note: it is possible in this case, but not easy. The book gives you the answer in the case of $a_0 = 2$, namely $a_n = \frac{4^n + 1}{4^n}$. Then they show you it is correct.

This form of "guess and check" is valid in mathematics, but requires cleverness. Not easy!

The other two are: (b) $a_0 = \frac{1}{2}$, $a_n = \frac{4^n - \frac{1}{2}}{4^n}$

(c) $a_0 = 10$, $a_n = \frac{4^n + 9}{4^n}$.

See a pattern??

For recursive sequences, there is an alternate strategy!

Look for fixed point populations. They sometimes act as limits for other sequences that start nearby....

Suppose $a_{n+1} = f(a_n)$ for some function where there is a fixed point population at a.

Then if we start the sequence at $a_0 = a$, then

$a_1 = f(a_0) = a, \quad a_2 = f(a_1) = f(a) = a, \quad \ldots$

and ~~[illegible]~~ the entire sequence satisfies

$$a = f(a)$$

same.

Solve this equation for values of a. These are your fixed pt populations.

<u>example</u> For $a_{n+1} = \frac{1}{4} a_n + \frac{3}{4}$, we solve

$a = \frac{1}{4} a + \frac{3}{4}$ to discover fixed pt pop.

$$a = \frac{1}{4} a + \frac{3}{4} \iff \frac{3}{4} a = \frac{3}{4} \iff a = 1.$$

Test it: Let $a_0 = 1$, $a_1 = \frac{1}{4}(1) + \frac{3}{4} = 1$. $a_2 = 1$, $a_3 = 1$, ...

So $a = 1$ is a fixed pt population.

Fact: If a recursively defined sequence converges, it must converge to a fixed pt population.

Go back to the 3 sequences we constructed:

$a_0 = 2$: $\{a_n\} = \{2, \frac{5}{4}, \frac{17}{16}, \frac{65}{64}, \cdots, \frac{4^n+1}{4^n}, \cdots\}$

$a_0 = \frac{1}{2}$: $\{a_n\} = \{\frac{1}{2}, \frac{7}{8}, \frac{31}{32}, \frac{127}{128}, \cdots, \frac{4^n - \frac{1}{2}}{4^n}, \cdots\}$

$a_0 = 10$: $\{a_n\} = \{10, \frac{17}{4}, \frac{25}{16}, \frac{73}{64}, \cdots, \frac{4^n+9}{4^n}, \cdots\}$.

By the fact above, if these sequences converge, they must converge to 1. They all do, though we won't construct the N for any choice of $\varepsilon > 0$ here.

Q: Can you conclude here that

$$\lim_{n\to\infty} \frac{4^n + c}{4^n} = 1 \quad \text{for any } c \in \mathbb{R} ?$$

The answer is yes. Think about it.

This gives us a strategy to analyze recursively defined sequences:

① Immediately find the fixed pt populations.

② If your sequence approaches any one of them, it will converge to it.

That fixed pt population becomes the limit.

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example Find $\lim_{n\to\infty} a_n$ if it exists, for

$a_{n+1} = \sqrt{5a_n}$, $a_0 = 2$.

Solution: Any fixed pt population will satisfy $a = \sqrt{5a}$. Solve this for a:

$a = \sqrt{5a} \iff a^2 = 5a \iff a^2 - 5a = 0 = (a-5)a.$

This is solved by $a = 0$, $a = 5$. Sequence is $\{a_n\} = \{2, \sqrt{10}, \sqrt[4]{250}, \sqrt[8]{6250}, \ldots\}$.
Hard to calculate, but sequence is increasing.
Hence by fact $\{a_n\} \to 5$.