

Class 7: ~~XXXXXXXXXX~~ Section 3.1

I

The idea of a limit of a function at a pt $x=c$ involves a very precise definition.

Calculus is the study of the properties of a function of one continuous variable. One of the fundamental properties is how a function varies as we vary the input variable.

The primary tool for studying how a function varies "at a pt" is the limit:

Let $f(x)$ be defined on an open interval containing a pt c , except possibly at c itself:

Def A function $f(x)$ as above is said to have a limit L at $x=c$ if for any choice of $\epsilon > 0$, there is a $\delta > 0$ so that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - c| < \delta.$$

Notes ① You will not be responsible for actually using this to "calculate" limits. However, you should understand it.

② If the limit does exist at a pt $x=c$, (then L is a real number), we say

$$\lim_{x \rightarrow c} f(x) = L$$

Notes cont'd.

③ Intuitive idea: Take a sequence of values for x that approach $x=c$.

ex. if $c=2$, then choose $1, 1.9, 1.99, 1.999, 1.9999, \dots$
as your infinite list of numbers (sequence)

Then see if the sequence of function values

$f(1), f(1.9), f(1.99), f(1.999), \dots$

approaches any particular number in \mathbb{R} .

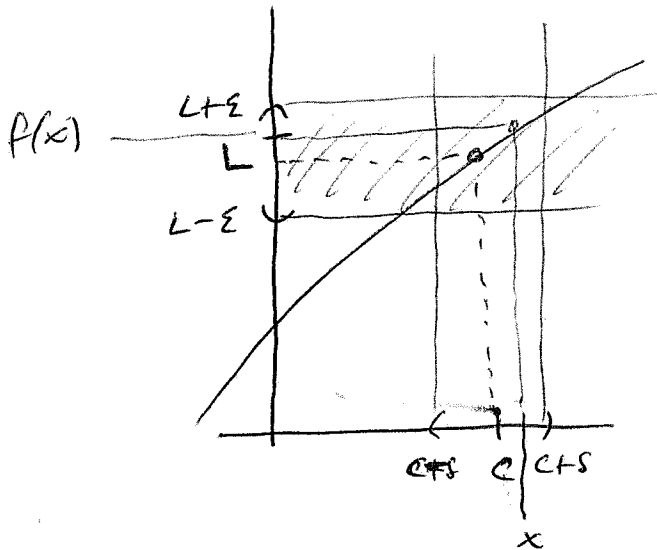
If it approaches a number L , and if every possible sequence you choose (on both sides of $x=c$) approaches L , then limit at $x=c$ of $f(x)$ exists and is L .

④ Notice that c does not have to be in the domain of f for the limit to exist at $x=c$ (in the definition, $x \neq c$!).

But the domain of f must be on both sides of $x=c$.

Notes cont'd

⑤ Visually, limits are easy to see:



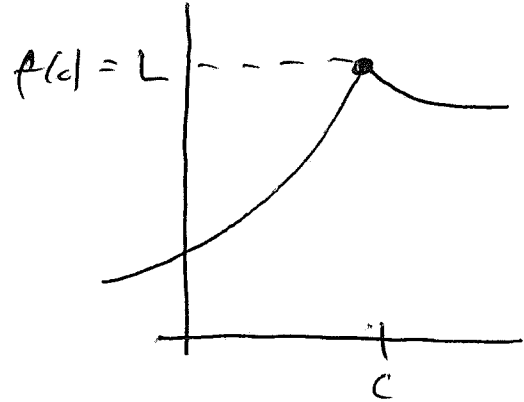
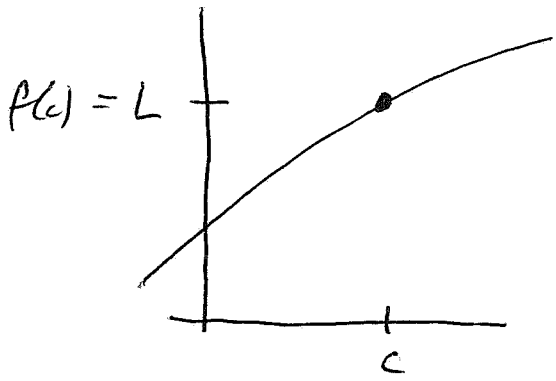
Given a small interval created using an $\epsilon > 0$, $(L-\epsilon, L+\epsilon)$ centered around L on vertical axis, can we create a small interval about $x=c$ on horiz. axis (except possibly at $x=c$), so

that for every δ at $x \in (c-\delta, c+\delta)$, its function value $f(x) \in (L-\epsilon, L+\epsilon)$. The answer in the above graph looks like yes, we can.

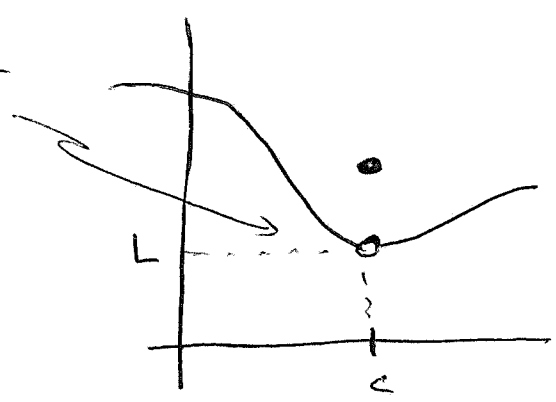
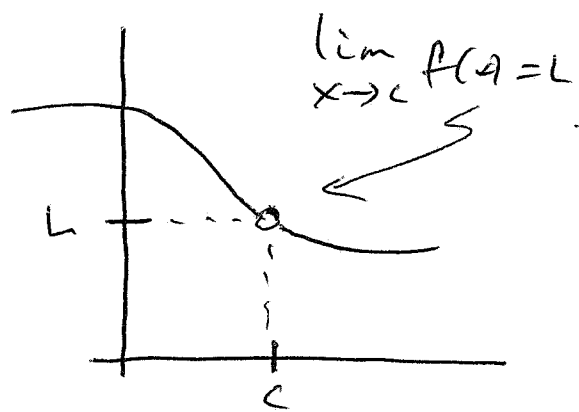
⑥ Visually, limit are easy to see: If a function graph looks like a smooth curve, possibly with a corner, ~~or~~ ~~or~~ smooth through $f(c)$, or if there is a hole in the graph only at what would be $f(c)$, then the limit will exist and be $f(c)$ or where the hole

Notes cont'd

⑦ Some visual examples:



If you can draw graph of $f(x)$ on an interval containing c without lifting your pencil from the paper, then $\lim_{x \rightarrow c} f(x) = L = f(c)$.

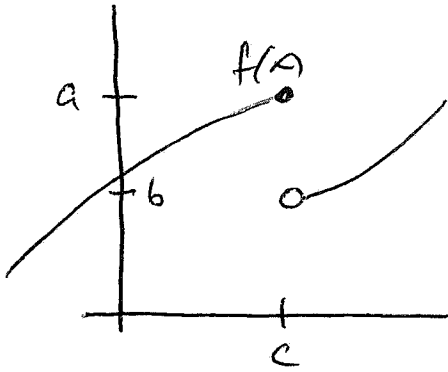


For a limit to exist of $f(x)$ at $x=c$, it doesn't matter what happens at $x=c$, only what happens "near" $x=c$ (on both sides).

Notes cont'd.

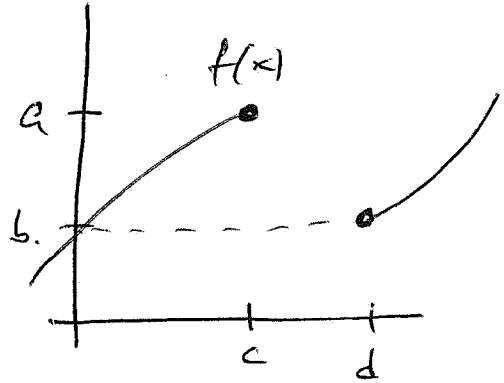
⑦ cont'd

Here $\lim_{x \rightarrow c^-} f(x) = a \neq b = \lim_{x \rightarrow c^+} f(x)$



Here $\lim_{x \rightarrow c^-} f(x) = a$

but $\lim_{x \rightarrow c^+} f(x)$ DNE



limit $f(x)$ as $x \rightarrow c$ exists and equals L

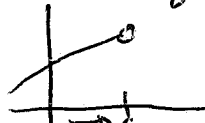
$\forall \epsilon$ $\lim_{x \rightarrow c^-} f(x) = L = \lim_{x \rightarrow c^+} f(x)$

On the left, if we approach $x=c$ from pts lower than c , then the function values "go to" a , but from above c , function values go to b . Since $a \neq b$, $\lim_{x \rightarrow c} f(x)$ does not exist

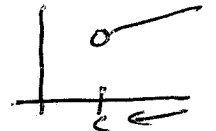
On the right, one cannot create a sequence of input pts x approaching $x=c$ from numbers larger than c . Hence $\lim_{x \rightarrow c} f(x)$ cannot exist.

One can talk about a limit from one of the 2 sides via

$\lim_{x \rightarrow c^-} f(x)$



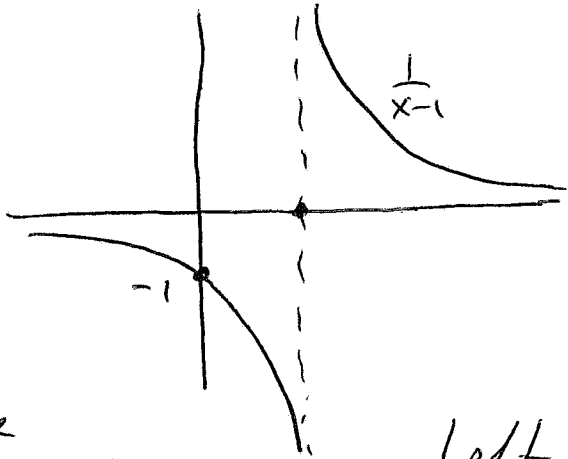
limit $\lim_{x \rightarrow c^+} f(x)$



Notes cont'd

⑧ How else can limits fail to exist?

example $\lim_{x \rightarrow 1} \frac{1}{x-1}$ does not exist.



What happens to the function values if we choose a sequence of pts x approaching $x=1$ from the

left? Try $\{1-.1, 1-.01, 1-.001, 1-.0001, \dots\}$

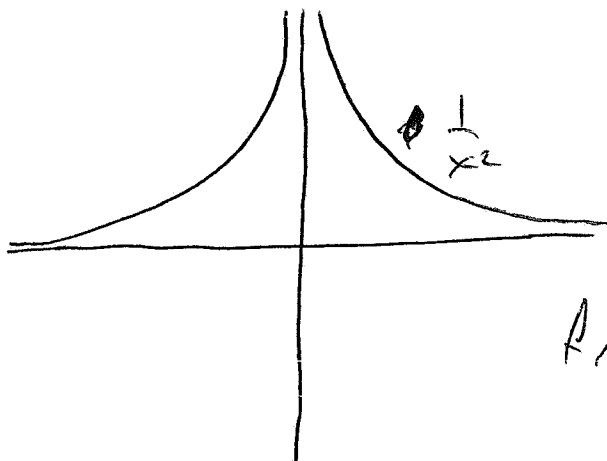
Here $\lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty$

from the right? Try $\{1.1, 1.01, 1.001, 1.0001, \dots\}$.

$\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty$

but neither exist

example $\lim_{x \rightarrow 0} \frac{1}{x^2}$ does not exist.



Test the 2 sides here again.

How does this case differ

from the last one?

Here $\lim_{x \rightarrow 0^-} f(x) = \infty = \lim_{x \rightarrow 0^+} f(x)$

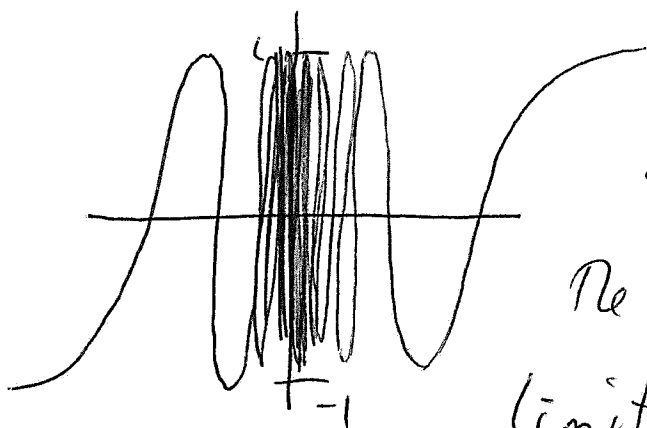
but neither exist

so no limit

Notes cont'd

⑧ cont'd

$\lim_{x \rightarrow 0} \sin \frac{\pi}{x}$ does not exist.



It is very hard to "see" the graph of this function near $x=0$.

The domain is all $x \neq 0$. But the limit may exist?

Choose the sequence $a_n = \frac{1}{10^n}$. Here $\{a_n\} \rightarrow 0$

~~the domain~~. The sequence of function values

$$\begin{aligned} \left\{ \sin \frac{\pi}{a_n} \right\} &= \{ \sin \pi, \sin 10\pi, \sin 100\pi, \sin 1000\pi, \dots \} \\ &= \{ 0, 0, 0, 0, 0, \dots \} \rightarrow 0. \end{aligned}$$

Hence it is possible that $\lim_{x \rightarrow 0} \sin \frac{\pi}{x} = 0$.

But if instead, we chose $b_n = \frac{2}{10^n + 1}$, then $\{b_n\} \rightarrow 0$

$$\begin{aligned} \text{And the sequence of function values } \left\{ \sin \frac{\pi}{b_n} \right\} &= \\ &= \left\{ \sin \frac{\pi}{2}, \sin \frac{11\pi}{2}, \sin \frac{101\pi}{2}, \sin \frac{1001\pi}{2}, \sin \frac{10001\pi}{2}, \dots \right\} \\ &= \{ 1, -1, 1, 1, 1, 1, 1, \dots \} \rightarrow 1. \end{aligned}$$

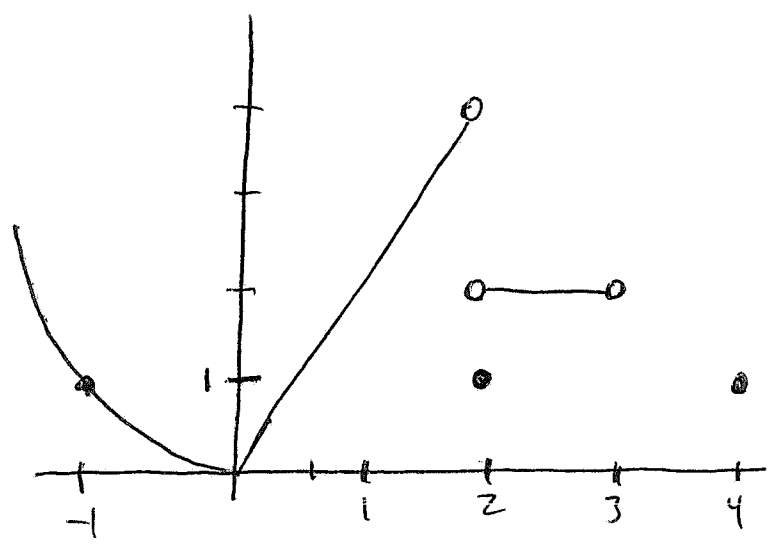
Notes cont'd.

⑧ cont'd

For a limit to exist, it must be unique (else the definition will fail). Hence

for $\sin(\frac{\pi}{x})$ there is no limit @ $x=0$.

⑨ For piecewise defined functions, most of the time it is only on the pts where 2 pieces of the domain meet where one needs to check for limit issues:



$$f(x) = \begin{cases} x^2 & x < 0 \\ 2x & 0 \leq x < 2 \\ 1 & x = 2 \\ 2 & x < 2 < 3 \\ 1 & x = 4 \end{cases}$$

At edges of pieces

$\lim_{x \rightarrow 0} f(x) = 0$

$\lim_{x \rightarrow 2} f(x)$ does not exist

$\lim_{x \rightarrow 3} f(x)$ does not exist

$\lim_{x \rightarrow 4} f(x)$ does not exist.