Class 9: Section 3.3.

We start with the continuity of compositions of functions.

Thm If $g(x)$ is continuous at $x=c$, with $g(c)=L$, and $f(x)$ is continuous at $x=L$, then $(f\circ g)(x)$ is continuous at $x=c$, and

$$\lim_{x\to c}(f\circ g)(x) = \lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$$
$$= f(g(c)) = f(L).$$

Notes ① Typically, one must work from the inside out when establishing continuity of a composition at a pt $x=c$.

ex. Where is $f(x)=\sin(\ln x)$ continuous?
Here the inside function $\ln x$ is cont on $(0,\infty)$, with range all $\mathbb{R}$. And since $\sin x$ is cont. on $\mathbb{R}$, $f(x)$ is cont on $(0,\infty)$.

ex. $g(x) = \ln(\sin x)$. Where is $g(x)$ cont?

Here, the inside function $\sin x$ is cont on all of $\mathbb{R}$, but its range is $[-1,1]$.

The outside function is cont. only on $(0,\infty)$

Hence only those input values whose range is in $(0,1] = [-1,1] \cap (0,\infty)$ are where $g(x)$ is continuous. Where is $\sin x \in (0,1]$?

$g(x)$ is continuous on the domain

$$\{x \in \mathbb{R} \mid \sin x \in (0,1]\}$$

$$= \{x \in \mathbb{R} \mid 0 < x \bmod 2\pi < \pi\}$$

Think about this one.

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ex. (7(c) pg 107)

Determine where $h(x) = \frac{1}{1+2x^{1/3}}$ is continuous.

$$\frac{1}{1+2x^{1/3}}$$

Note: The book uses $g(x) = x^{1/3}$ and $f(x) = \frac{1}{1+2x}$ to write $h(x) = f(g(x)) = (f \circ g)(x)$. The result is $h(x)$ is continuous on $\{x \in \mathbb{R} \mid x \neq -\frac{1}{8}\}$.

Here, we split $h(x)$ using $f(x) = \frac{1}{x}$ and $g(x) = 1+2x^{1/3}$ so that $h(x) = f(g(x)) = f(1+2x^{1/3}) = \frac{1}{1+2x^{1/3}}$.

Here, $g(x) = 1+2x^{1/3}$ is continuous everywhere, and its range is all of $\mathbb{R}$. But $f(x)$ is not defined when $x=0$, so we need to rule out all input values for $g(x)$ where the output value is 0: rule out all $x \in \mathbb{R}$ where $g(x) = 0$.

$g(x) = 0$ when $1+2x^{1/3} = 0$, or $x^3 = -\frac{1}{2}$, or $x = -\frac{1}{8}$.

Hence $h(x) = f(g(x)) = \frac{1}{1+2x^{1/3}}$ is continuous on the domain $\{x \in \mathbb{R} \mid x \neq -\frac{1}{8}\} = (-\infty, -\frac{1}{8}) \cup (-\frac{1}{8}, \infty)$.

Recall a sequence $\{a_n\}_{n=1}^{\infty}$ has a limit L, and we say $\lim_{n\to\infty} a_n = L$, if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ so that

$$|a_n - L| < \varepsilon \text{ whenever } n > N.$$

We used this to show $\lim_{n\to\infty} \frac{1}{n} = 0$: Given any $\varepsilon > 0$, choose $N \geq \frac{1}{\varepsilon}$. (can you see this?)

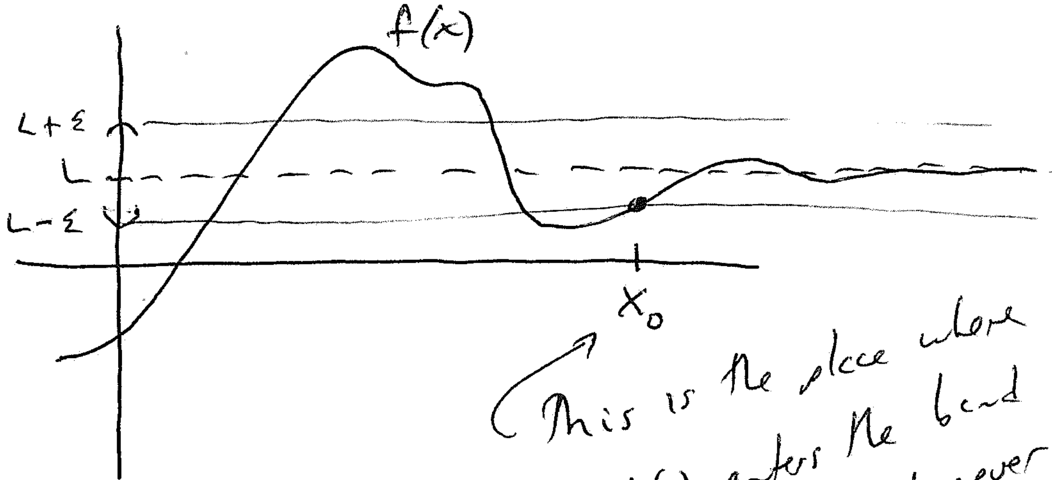
This also works for functions of a continuous variable $x \in \mathbb{R}$:

A function $f(x)$ has a limit L at infinity, denoted $\lim_{x\to\infty} f(x) = L$, if for every $\varepsilon > 0$ there is an $x_0 \in \mathbb{R}$, so that

$$|f(x) - L| < \varepsilon \text{ whenever } x > x_0$$

Recognize that this is a formal, precise definition of a limit at infinity.

Visually,

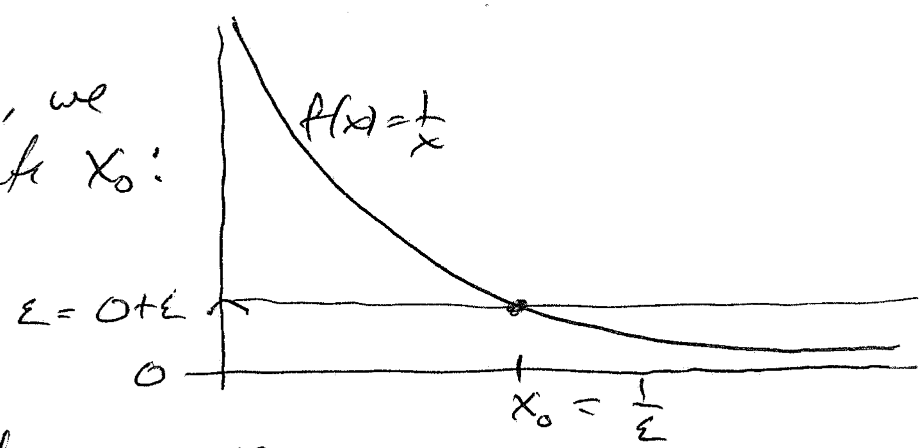


Can be used to show

$\lim_{x\to\infty} \frac{1}{x} = 0$. Given $\varepsilon>0$, we seek

a pt $x_0 \in \mathbb{R}$ when function $f(x)$ will always lie between 0 and ε ever afterward.

Given an $\varepsilon>0$, we can calculate x_0:



Want

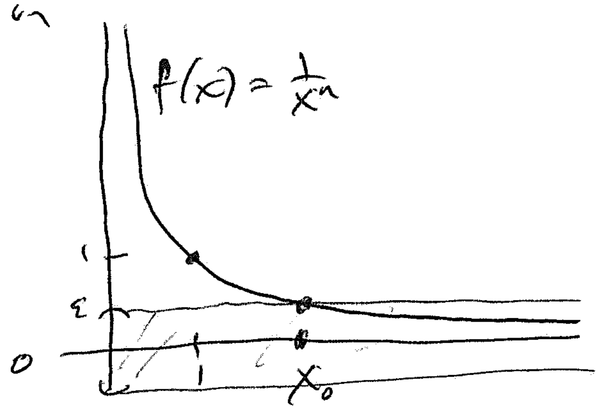
$|f(x)-L|<\varepsilon$ whenever $x>x_0$.

$\left|\frac{1}{x}-0\right|<\varepsilon \implies \frac{1}{x}<\varepsilon$. This will be true when $x>\frac{1}{\varepsilon}$.

So choose $x_0=\frac{1}{\varepsilon}$. Then def of limit holds.

ex. Show $\lim_{x\to\infty} \frac{1}{x^n} = 0$, for $n \in \mathbb{N}$.

<u>Method 1</u>: By definition



Want $|f(x) - L| < \varepsilon$

when $x > x_0$.

Choose $L = 0$.

$|f(x) - L| = \left|\frac{1}{x^n} - 0\right| = \frac{1}{x^n}$

and $\frac{1}{x^n} < \varepsilon$, precisely when $\sqrt[n]{\frac{1}{\varepsilon}} < x$.

choose $x_0 = \sqrt[n]{\frac{1}{\varepsilon}}$. Then def holds. and

$\lim_{x\to\infty} \frac{1}{x^n} = 0$.

<u>Method 2</u>: Use limit Laws: (and previous result)

$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to\infty} \left(\frac{1}{x}\right)^n = \lim_{x\to\infty} \overbrace{\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)\cdots\left(\frac{1}{x}\right)}^{n \text{ times}}$$

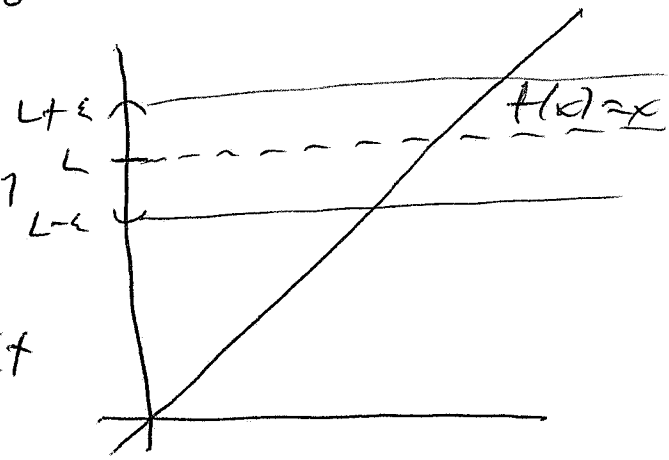
$$\underset{\text{for limits}}{\overset{\text{Product Rule}}{=}} \left(\lim_{x\to\infty} \frac{1}{x}\right)\left(\lim_{x\to\infty} \frac{1}{x}\right)\cdots\left(\lim_{x\to\infty} \frac{1}{x}\right)$$

as long as each of these exists.

They do, and

$$= 0 \cdot 0 \cdot \cdots \cdot 0 = 0.$$

ex. Calculate $\lim_{x \to \infty} x$, if it exists.



Suppose you choose any real number $L > 0$ as a possible limit

Then for any choice of $\varepsilon > 0$, once the function input values get larger than $L + \varepsilon$, $f(x) = x$ will leave the band and never return. Hence there is no real number L that can serve as the limit

Hence limit does not exist.

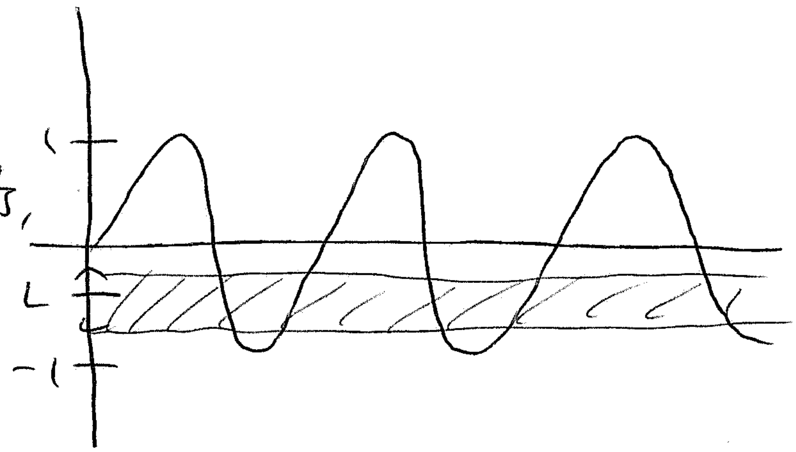
Note: This kind of nonexisting limit is particular. We say $\lim_{x \to \infty} x = \infty$.

This is different from $\lim_{x \to \infty} \sin x$, which also does not exist.

See box pg 95 and next example.

ex. Calculate $\lim_{x\to\infty} \sin x$, if it exists.

It would make no sense, if L exists, that $L>1$ or $L<-1$ (why?).



So choose some $L\in[-1,1]$ to possibly serve as the limit. Choose an $\varepsilon>0$ small enough, like in the picture. Can you see that there will be no $x_0\in\mathbb{R}$ where for all $x>x_0$, $|(\sin x)-L|<\varepsilon$??

$\lim_{x\to\infty}\sin x$ does not exist.

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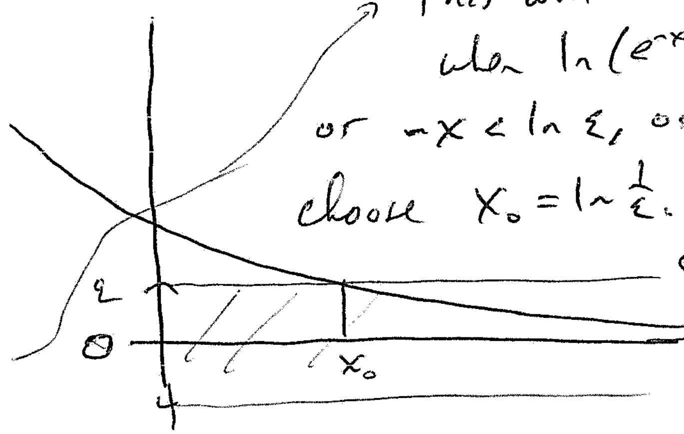
ex. Show $\lim_{x\to\infty} e^{-x}=0$.

Like $\frac{1}{x}$, we seek a pt $x_0\in\mathbb{R}$ where when $x>x_0$,

$|f(x)-L|=|e^{-x}-0|=e^{-x}<\varepsilon$.

This will be true when $\ln(e^{-x})<\ln\varepsilon$ or $-x<\ln\varepsilon$, or $x>\ln\frac{1}{\varepsilon}$. choose $x_0=\ln\frac{1}{\varepsilon}$. Then def will hold.



Note: ① Polynomials (of positive degree) never have limits at infinity or $-\infty$ (why not?)

② Rational functions sometimes do and sometimes do not. It turns out there is an easy way to tell!!

Thm For $f(x) = \frac{p(x)}{q(x)}$ rational, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg(p) < \deg(q) \\ L & \text{if } \deg(p) = \deg(q) \\ \infty & \text{if } \deg(p) > \deg(q) \end{cases}$$

where L is a real number. (what is L?)

Note: The proof is not difficult and kinda interesting. The next example should give you and idea of how it would work.

ex. Show $\lim\limits_{x\to\infty} \dfrac{3x^2-4x+5}{7x^3+3x^2+6} = 0.$

Strategy: Use the limit Laws and a little manipulation of the rational function to calculate the limit directly.

Solution: The previous theorem establishes this since $\deg(p) = \deg(3x^2-4x+5) = 2$. and $\deg(q) = \deg(7x^3+3x+6) = 3$.

Since $f(x) = \dfrac{3x^2-4x+5}{7x^3+3x^2+6}$ is continuous on $(0,\infty)$ (why?), we know it is possible for the limit to exist. We cannot apply the Quotient Law for limits directly since $\lim\limits_{x\to\infty} p(x) = \lim\limits_{x\to\infty} 3x^2-4x+5$ does not exist.

ex (cont'd)

But, using a common trick in math; multiplication by a "clever form of one", we can manipulate $f(x)$ so that we can use the Quotient Rule for limits:

$1 = \left(\frac{\frac{1}{x^3}}{1/x^3}\right)$ is continuous on $(0,\infty)$, and

thus so is

$$\frac{(3x^2-4x+5)}{(7x^3+3x^2+6)}\left(\frac{1/x^3}{1/x^3}\right) = \frac{\frac{3}{x} - \frac{4}{x^2} + \frac{5}{x^3}}{7 + \frac{3}{x} + \frac{6}{x^3}}$$

since it is the product of 2 continuous functions on $(0,\infty)$. Hence

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2-4x+5}{7x^3+3x^2+6} = \lim_{x\to\infty} \frac{(3x^2-4x+5)\left(\frac{1}{x^3}\right)}{(7x^3+3x^2+6)\left(1/x^3\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{3}{x} - \frac{4}{x^2} + \frac{5}{x^3}}{7 + 3/x + 6/x^3}.$$

ex. (cont'd) Now for $f(x) = \frac{p(x)}{q(x)} = \frac{\frac{3}{x} - \frac{4}{x^2} + \frac{5}{x^3}}{7 + \frac{3}{x} + \frac{6}{x^3}}$.

Now, since $\lim_{x \to \infty} p(x) = \lim_{x \to \infty} \left(\frac{3}{x} - \frac{4}{x^2} + \frac{5}{x^3} \right)$

$$\overset{\text{sum rule}}{=} \lim_{x \to \infty} \frac{3}{x} + \lim_{x \to \infty} \frac{-4}{x^2} + \lim_{x \to \infty} \frac{5}{x^3}$$

$$\overset{\text{const mult rule}}{=} 3\left(\lim_{x \to \infty} \frac{1}{x}\right) - 4\left(\lim_{x \to \infty} \frac{1}{x^2}\right) + 5\left(\lim_{x \to \infty} \frac{1}{x}\right.$$

$$= 3(0) - 4(0) + 5(0) = 0$$

and $\lim_{x \to \infty} q(x) = \lim_{x \to \infty} \left(7 - \frac{3}{x} + \frac{6}{x^3}\right) = 7 - 3(0) + 6(0)$

by the same reasoning, we get

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{p(x)}{q(x)} = \frac{\lim_{x \to \infty} p(x)}{\lim_{x \to \infty} q(x)} = \frac{0}{7} = 0$$

by the Quotient Rule for limits.

Can you see that this "trick" (multiplication by a "clever form of 1", can be used in the proof of the theorem?