Class 10: Section 3.4

Last class we ~~used~~ stated the theorem useful for studying a property of rational functions:

<u>Thm</u> Let $f(x) = \frac{p(x)}{q(x)}$ be rational. Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg(p) < \deg(q) \\ L & \text{if } \deg(p) = \deg(q) \\ \infty & \text{if } \deg(p) > \deg(q) \end{cases}$$

where L is a real number.

We then used this to ~~solve~~ calculate

$$\lim_{x\to\infty} \frac{3x^2-4x+5}{7x^3+3x^2+6} = 0$$

(Recall the multiplication of 1 in the form $\left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right)$.).

exercise: Do the same calculation for $h(x) = \frac{3x^3-4x+5}{7x^3+3x^2+6}$.

Here $\deg(p) = \deg(3x^3-4x+5) = 3$

$\deg(q) = \deg(7x^3+3x^2+6) = 3$.

Strategy: Use a clever multiplication by 1 and the limit laws to evaluate.

Solution: $h(x)$ is certainly continuous for x large enough. Hence

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty}\frac{3x^3-4x+5}{7x^3+3x^2+6} = \lim_{x\to\infty}\frac{3x^3-4x+5}{7x^3+3x^2+6}\left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right)$$

(why $\frac{1}{x^3}$?)

$$= \lim_{x\to\infty}\frac{3-\frac{4}{x^2}+\frac{5}{x^3}}{7+\frac{3}{x}+\frac{6}{x^3}}$$

$$\overset{\text{quotient Law for limits}}{=} \frac{\lim_{x\to\infty}\left(3-\frac{4}{x^2}+\frac{5}{x^3}\right)}{\lim_{x\to\infty}\left(7+\frac{3}{x}+\frac{6}{x^3}\right)}$$

$$\overset{\text{Constant sum multiple Rule for limits}}{=} \frac{\lim_{x\to\infty} 3 - 4\left(\lim_{x\to\infty}\frac{1}{x^2}\right)+5\left(\lim_{x\to\infty}\frac{1}{x^3}\right)}{\lim_{x\to\infty} 7 + 3\left(\lim_{x\to\infty}\frac{1}{x}\right)+6\left(\lim_{x\to\infty}\frac{1}{x^3}\right)}$$

$$\boxed{\text{Note: } \lim_{x\to\infty}\frac{1}{x^2} = \left(\lim_{x\to\infty}\frac{1}{x}\right)\left(\lim_{x\to\infty}\frac{1}{x}\right) \text{ by Product Rule for limits}}$$

$$= \frac{3-4(0)+5(0)}{7+3(0)+6(0)} = \frac{3}{7} = L \text{ in thm.}$$

Can you see what the L is in the theorem?

In thm, if $\deg(p) = \deg(q)$ for $h(x) = \frac{p(x)}{q(x)}$

then $\lim_{x \to \infty} h(x) = L$, where L is the ratio of the leading coefficients of p, q.

ex. Now do the same calculation on

~~ex~~ $\lim_{x \to \infty} i(x)$, where $i(x) = \frac{3x^7 - 4x + 5}{7x^3 + 3x^2 + 6}$.

to show limit does not exist.

Other ways to see limits at infinity of rational functions?

(I) The third type in the ~~system~~ theorem, where $\deg(p) > \deg(q)$, is called an improper rational function. One can always write an improper rational function as the sum of a polynomial and a proper rational function (where $\deg(p) < \deg(q)$) via long division:

ex. (pg III, middle)

$$\frac{x^4+2x-5}{x^2-x+2} = x^2+x-1 - \frac{x+3}{x^2-x+2}$$

How so? left hand side is

$$\begin{array}{r|l} & x^2+x-1 \\ \hline x^2-x+2 & x^4+2x-5 \\ & -(x^4-x^3+2x^2) \\ \hline & x^3-2x^2+2x \\ & -(x^3-x^2+2x) \\ \hline & -x^2-5 \\ & -(-x^2+x-2) \\ \hline & -x-3 \end{array}$$

result is x^2+x-1 with remainder $-x-3 = -(x+3)$.

So right hand side is

$$x^2+x-1 - \frac{(x+3)}{x^2-x+2}.$$

Can you see why

$$\lim_{x\to\infty} \frac{x^4+2x-5}{x^2-x+2} = \lim_{x\to\infty} x^2+x-1 - \frac{x+3}{x^2-x+2}$$ Does not exist?

(II) The theorem is really a statement on the relationship between x^m and x^n the 2 power functions that make up the leading monomials of $p(x)$, $q(x)$.

Recall problem 26, Section 1.2:

for $n > m$, $x^n > x^m$ when $x > 1$

But this means $\lim_{x \to \infty} \frac{x^n}{x^m} = \lim_{x \to \infty} x^{n-m}$

where $n-m = r > 0$. Here limit DNE.

And if $n < m$, $\lim_{x \to \infty} \frac{x^n}{x^m} = \lim_{x \to \infty} x^{n-m}$

where $n-m = r < 0$. Here limit is 0.

Notes: ① If $n = m$, $\lim_{x \to \infty} \frac{x^n}{x^m} = 1$ and

$$\lim_{x \to \infty} \frac{a x^n}{b x^n} = \frac{a}{b} \lim_{x \to \infty} \frac{x^n}{x^n} = \frac{a}{b}.$$

② As x gets large (going to ∞), all of the lower order monomials become inconsequential and do not contribute to the determination of the limit.

why?

Here is a new problem: Let $g(x) = \frac{\sin x}{x}$

Find: (a) $\lim_{x \to \infty} g(x)$

(b) $\lim_{x \to 0} g(x)$

if they exist.

Notes: ① A chart or graph may be helpful here but may be misleading

② $g(x)$ is continuous on all of $\{x \in \mathbb{R} \mid x \neq 0\}$.

as the product of 2 continuous functions on this domain.

③ Direct calculations will be tough.

But we can compare $g(x)$ to functions we know the limit of!

<u>Sandwich Thm</u>

(I) If $f(x) \leq g(x) \leq h(x)$ are continuous funct. for all x on an open interval containing a pt c (except possibly at $x=c$), and

$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$.

Sandwich Thm (cont'd)

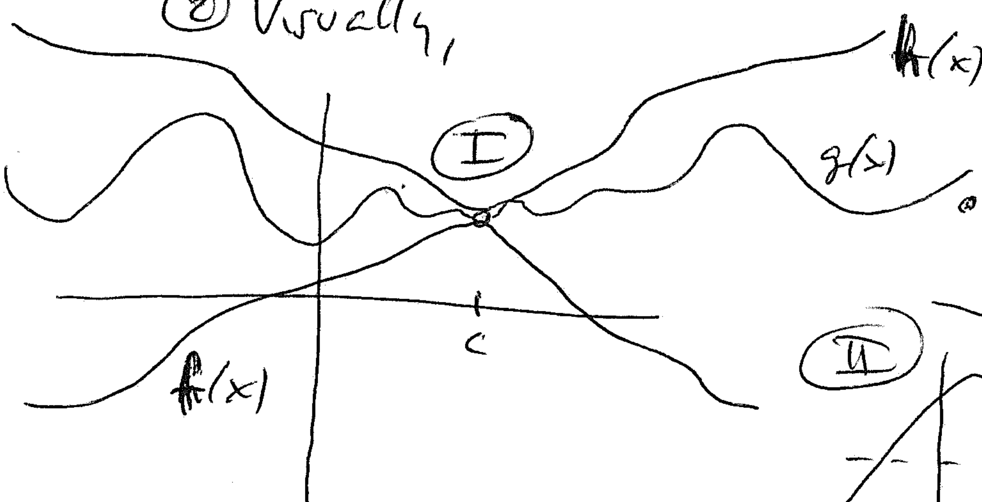
(II) if $f(x) \le g(x) \le h(x)$ are continuous on some interval $[x_0, \infty)$ and

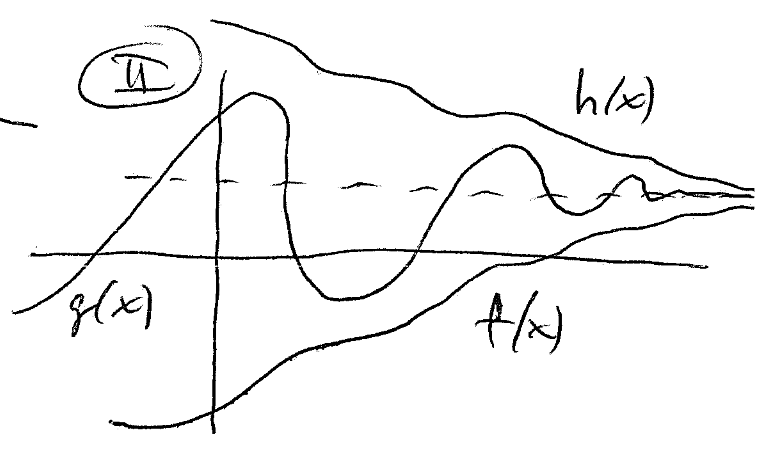
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = L, \text{ then } \lim_{x\to\infty} g(x) = L.$$

(III) Same for $\lim_{x\to -\infty} g(x)$.

Notes ① in effect, the functions $f(x)$ and $h(x)$ squeeze or sandwich $g(x)$, forcing it to have the same limit.

② Visually,





③ No need for any direct calculations

Back to example

② Find $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{\sin x}{x}$ if it exists.

First, notice that for all $x > 0$, (for all $x \in \mathbb{R}$ really)

$$-1 \leq \sin x \leq 1$$

and dividing by any positive x doesn't change the sense of the inequality:

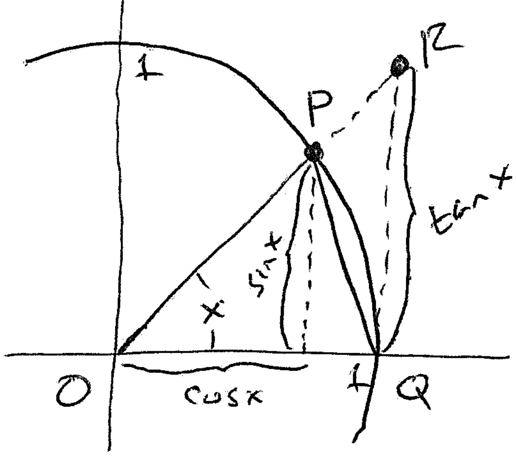
$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

So let $f(x) = -\frac{1}{x}$, $h(x) = \frac{1}{x}$. Then part II of the Sandwich Thm holds and since

$$\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to\infty} -\frac{1}{x} = 0$$

it follows that $\boxed{\lim_{x\to\infty} \frac{\sin x}{x} = 0.}$

For part ⓑ, find $\lim_{x \to 0} \frac{\sin x}{x}$ if it exists, we need to be more clever:



Let P be a point in the first quadrant on the unit circle. P has coordinates $(\cos x, \sin x)$ where x is the angle the ray $\overrightarrow{OP}$ makes with the positive horizontal axis.

Verify that $\text{area}(\Delta OPQ) \leq \text{area}(\text{sector } OPQ) \leq \text{area}(\Delta ORQ)$ for any $x \in (0, \frac{\pi}{2})$. But then

$$\tfrac{1}{2}(1)(\sin x) \leq \pi (1)^2 \left(\tfrac{x}{2\pi}\right) \leq \tfrac{1}{2}(1)(\tan x)$$

or

$$\sin x \leq x \leq \tan x$$

when we cancel out the $\frac{1}{2}$ and the π's.

Divide by the positive quantity $\sin x$ (for $x \in (0, \frac{\pi}{2})$) to get

$$1 \leq \frac{x}{\sin x} \leq \frac{\tan x}{\sin x} = \frac{1}{\cos x}$$

And invert to get (reversing the sense of the inequality)

$$\cos x \leq \frac{\sin x}{x} \leq 1 .$$

Now the middle of the inequality is $g(x)$.
Let $H(x) = \cos x$ and $h(x) = 1$. Both of these are continuous on all $\mathbb{R}$, and since

$$\lim_{x\to 0} \cos x = \lim_{x\to 0} 1 = 1,$$

by the Squeezing Thm, we get (Sandwich

$$\boxed{\lim_{x\to 0^+} \frac{\sin x}{x} = 1.}$$

Note: ① The hard part is to come up with the comparison functions $H(x)$ and $h(x)$. But do not worry, here you will not be asked to be too clever.

② To actually establish that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, (the full limit), one would have to do the same thing for the "other side" of 0, for $x<0$, small negative angles.