

Class 11: ~~XXXXXXXXXX~~ Section 3.5 I

There are other techniques for calculating limits of functions:

(I) Inspection of continuity: If you can discern that the function  $f(x)$  is continuous at  $x=c$ , then by continuity

$$\lim_{x \rightarrow c} f(x) = f(c).$$

You can call this "plugging in the value".

ex: Calculate  $\lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x}$ , if possible.

Strategy: Check for continuity. If cont. at  $x=0$ , then plug in value and evaluate.

Solution: Both  $\sin x$  and  $1 + \cos x$  are continuous functions on all of  $\mathbb{R}$ . And since  $1 + \cos(0) \neq 0$ , near  $x=0$ , the ratio  $\frac{\sin x}{1 + \cos x}$  is also cont as the quotient of 2 continuous functions.

Solution cont'd.

Thus  $\frac{\sin x}{1 + \cos x}$  is cont at  $x=0$  and

$$\lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} = \frac{\sin(0)}{1 + \cos(0)} = \frac{0}{1+1} = 0. \quad \square$$

Note: The function  $\frac{\sin x}{1 + \cos x}$  is not continuous on all of  $\mathbb{R}$ , though. In fact, the largest interval containing  $x=0$  where  $\frac{\sin x}{1 + \cos x}$  is continuous is  $(-\pi, \pi)$ . (Why?)

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② More "clever multiplication by 1"

(Manipulation of a function in a way that doesn't change the function, but makes it easier to work with.)

ex. Show  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$ .

example (cont'd).

Strategy: Use the same "clever multiplication by 1" trick we used with rational functions to allow the Limit Rules to apply.

Solution: The function  $f(x) = \frac{1 - \cos x}{x}$  is not continuous at  $x=0$ . Hence we cannot "plug in"  $x=0$ . But it is cont "near 0". Hence the limit may exist.

We try a "clever mult by 1" via mult. of the numerator by its conjugate:

For an expression involving 2 terms like  $a+b$ , the conjugate expression is  $a-b$ . Multiplying an expression by its conjugate always yields the difference of its squares:

$$(a+b)(a-b) \stackrel{\text{FOIL}}{=} a^2 + ab - ab - b^2 = a^2 - b^2$$

Solution cont'd.

Why mult by a conjugate expression?

It is a form of manipulation that has use: Like when rationalizing a denominator

$$\text{ex. } \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\text{ex. } \frac{4}{\sqrt{2}-\sqrt{3}} = \frac{4}{\sqrt{2}-\sqrt{3}} \left( \frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}} \right) = \frac{4(\sqrt{2}+\sqrt{3})}{2+3} = \frac{4}{5}(\sqrt{2}+\sqrt{3}).$$

For us:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cancel{\sin x}}{\cancel{1+\cos x}} \frac{1-\cos x}{x} &= \lim_{x \rightarrow 0} \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} \\ &= \lim_{x \rightarrow 0} \frac{1-\cos^2 x}{x(1+\cos x)} \end{aligned}$$

This only looks more complicated, yet

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1+\cos x)} \\ &= \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right) \left( \frac{\sin x}{1+\cos x} \right). \end{aligned}$$

Both of the limits of the 2 fractions in parentheses exist by previous examples, hence by the Product Rule for limits

Solution cont'd

$$\lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right) \left( \frac{\sin x}{1 + \cos x} \right) \stackrel{\substack{\text{Prod} \\ \text{Rule} \\ \text{for} \\ \text{limits.}}}{=} \underbrace{\left( \lim_{x \rightarrow 0} \frac{\sin x}{x} \right)}_1 \underbrace{\left( \lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} \right)}_0$$

$$= 1 \cdot 0 = 0.$$

Hence  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$ . □

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### III Change of variable

ex. Calculate  $\lim_{x \rightarrow 0} 3x$ . Here  $3x$  is a polynomial and continuous at  $x=0$ . Hence  $\lim_{x \rightarrow 0} 3x = 0$ .

Suppose a function  $h(x)$  can be written as a composition of 2 other functions  $h(x) = f(g(x))$ . Then it is often possible to change the variable in  $h(x)$  to make the function easier to study:

Let  $z = g(x)$ , the inside function. Then  $h(x) = f(g(x)) = f(z)$ .

III cont'd.

If the new variable (as a function)

$z = g(x)$  is continuous at  $x=c$ ,

$$\lim_{x \rightarrow c} z = \lim_{x \rightarrow c} g(x) = g(c) = L, \text{ then}$$

$$\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} f(g(x)) = \lim_{z \rightarrow L} f(z)$$



we have changed the variable in the limit.

Many times the limit on the right is easier to calculate.

ex. Show  $\lim_{x \rightarrow 0} \frac{\sin 3x}{5x}$  exists and calculate.

Strategy: This limit looks like  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$  which does exist. Use a change of variable to make it look like  $\frac{\sin x}{x}$  and then calc.

Solution: The numerator in  $h(x) = \frac{\sin 3x}{5x}$

looks like a composition  $f \circ g$  where  $f(x) = \sin x$  and  $g(x) = 3x$ .

Let  $z = 3x$ . Since  $\lim_{x \rightarrow 0} z = \lim_{x \rightarrow 0} 3x = 0$ ,

we change the variable:

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{5x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{\frac{5}{3}(3x)} = \lim_{z \rightarrow 0} \frac{\sin z}{\frac{5}{3}z}$$

$$\begin{array}{l} \text{const} \\ \text{multiple} \\ \text{rule} \\ \text{for} \\ \text{limits.} \end{array} \quad \frac{3}{5} \lim_{z \rightarrow 0} \frac{\sin z}{z} = \frac{3}{5}(1) = \frac{3}{5}$$

Hence  $\lim_{x \rightarrow 0} \frac{\sin 3x}{5x}$  exists and equals  $\frac{3}{5}$ .

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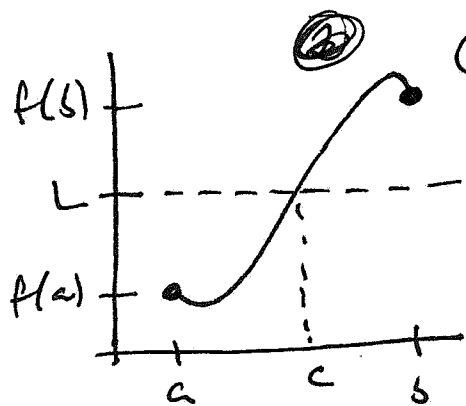
## Section 3.5 Properties of Continuous Fns.

One of the more profound and obvious theorems of calculus relies solely on the continuity of a function on a closed interval:

## Intermediate Value Theorem (IVT)

Let  $f(x)$  be continuous on  $[a, b]$ . If  $L \in \mathbb{R}$  where either  $f(a) < L < f(b)$  or  $f(b) < L < f(a)$ , then there is at least one  $x \in (a, b)$ , where  $f(x) = L$ .

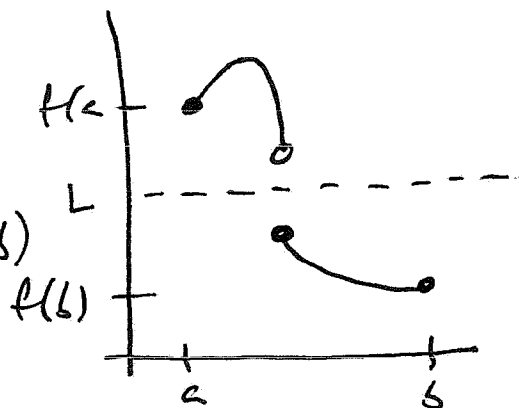
Notes ① In other words, a continuous function achieves ALL of its intermediate values on a closed interval.



② This is an example of an existence theorem. It does not and cannot help to actually find a value of  $c$ , where  $f(c) = L$ . It only establishes its existence.

③ Without continuity, the theorem fails.

Here there is no  $x = c \in (a, b)$  where  $f(c) = L$ .





example Show  $f(x) = 8x^5 - 4x^4 + 3x - 5$  has a root (an input value  $c$ , where  $f(c) = 0$ ) on the interval  $[0, 1]$ .

Strategy Use the Intermediate Value Thm.

Solution:  $f(x)$  is a polynomial and hence is continuous everywhere. So  $f(x)$  is continuous on  $[0, 1]$ . At the end pts,  
 $f(0) = 8(0)^5 - 4(0)^4 + 3(0) - 5 = -5 < 0$   
 $f(1) = 8(1)^5 - 4(1)^4 + 3(1) - 5 = 2 > 0$

Since  $f(x)$  is cont on  $[0, 1]$  and

$$-5 = f(0) < 0 < f(1) = 2$$

By the IVT, we know there must exist a pt  $c \in (0, 1)$  where  $f(c) = 0$

(0 was our choice of  $L$ ). Here  $c$  is a root of  $f(x)$ .

