

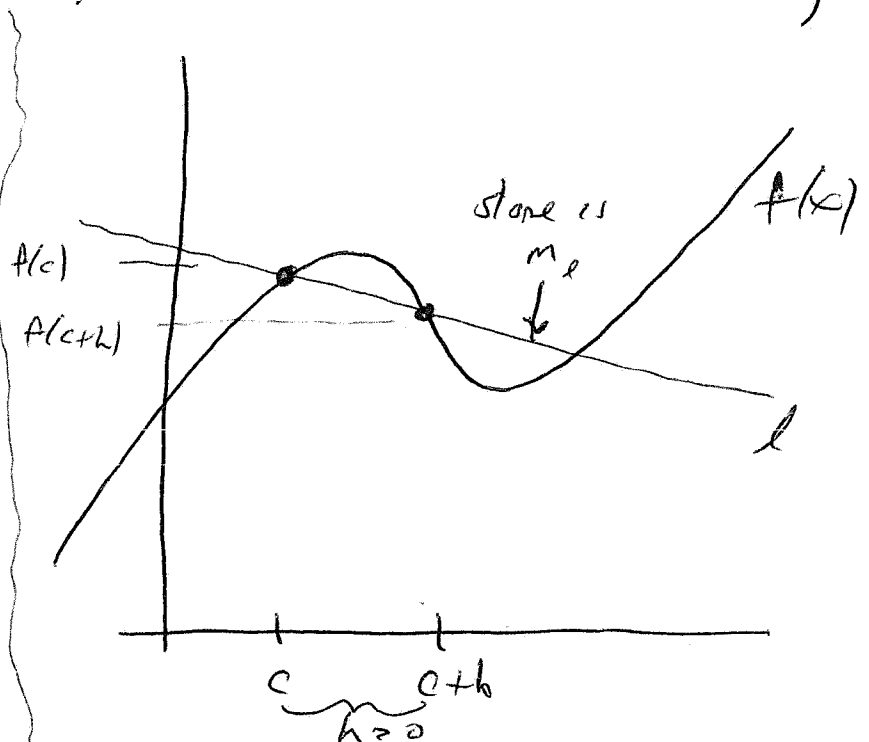
Class 13: ~~Section 4.1~~ Section 4.1 I

Last class, we defined a way to construct a tangent line to the graph of $f(x)$ at the point $x=c$ in its domain:

Choose a small number $h > 0$ (when we want our definitions and/or constructions to utilize a small number but we do not want to lock ourselves in to any specific choice, we use a parameter like h or ϵ .)

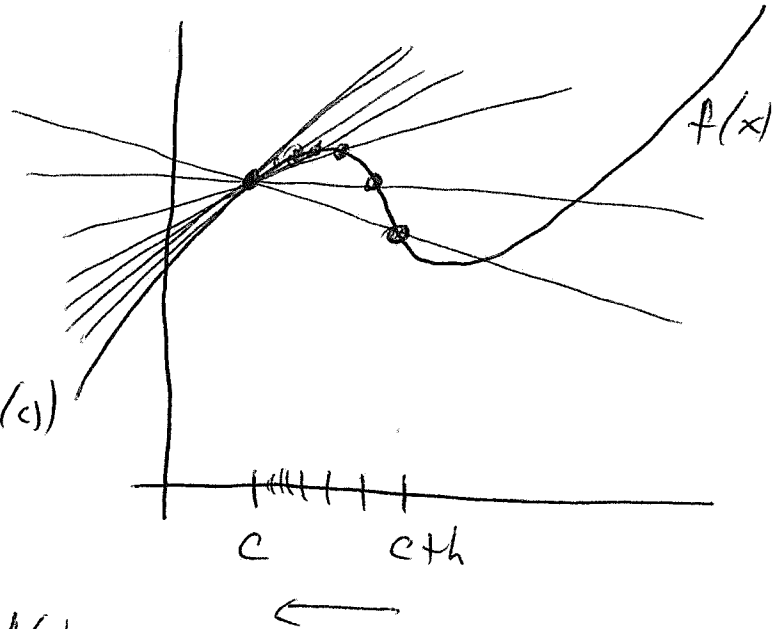
Then for the line through $(c, f(c))$ and $(c+h, f(c+h))$, its slope is

$$\begin{aligned} m_p &= \frac{f(c+h) - f(c)}{(c+h) - c} \\ &= \frac{f(c+h) - f(c)}{h} \end{aligned}$$



If we then "push" h to 0, we get

for each choice of smaller h 's going to 0, the line joining $(c, f(c))$ and $(c+h, f(c+h))$



has slope $\frac{f(c+h) - f(c)}{h}$,

so that in the limit, as h goes to 0,

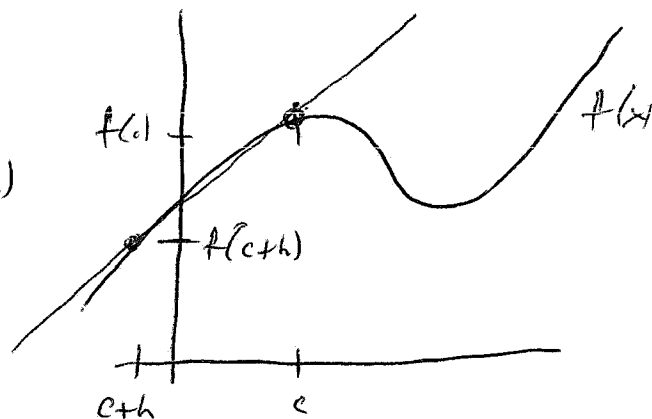
$$\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$$

If this right side limit exists, then the slopes "go" to the slope of a unique line.

Do the same for the other side (small $h < 0$).

to get

$$\lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h}$$



$$\text{If } \lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$$

and both exist, then the full limit

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

exists.

Def The derivative of f with respect to x , at $x=c$, when it exists, is denoted

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

Notes ① If $f'(c)$ exists, we say f is differentiable at $x=c$.

② The quantity $\frac{f(c+h) - f(c)}{h}$ for any choice of h is called a difference quotient

Sometimes it is denoted $\left. \frac{\Delta f}{\Delta x} \right|_{x=c}$

$$\frac{\Delta f}{\Delta x} = \frac{\text{change in } f}{\text{change in } x} = \frac{f(c+h) - f(c)}{(c+h) - c}$$

② cont'd

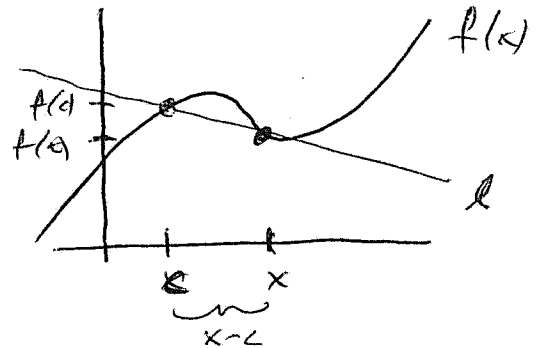
The difference quotient $\left. \frac{\Delta f}{\Delta x} \right|_{x=c}$ is called the average rate of change of f on Δx .

And $f'(c)$ is called the instantaneous rate of change of f at $x=c$.

③ Another way to define the limit is to not bother with the h :

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

↖
slope of l



④ More Notation

② For $y = f(x)$, we sometimes write $y(x)$.

Then derivative is $y'(x)$.

③ Leibniz notation:

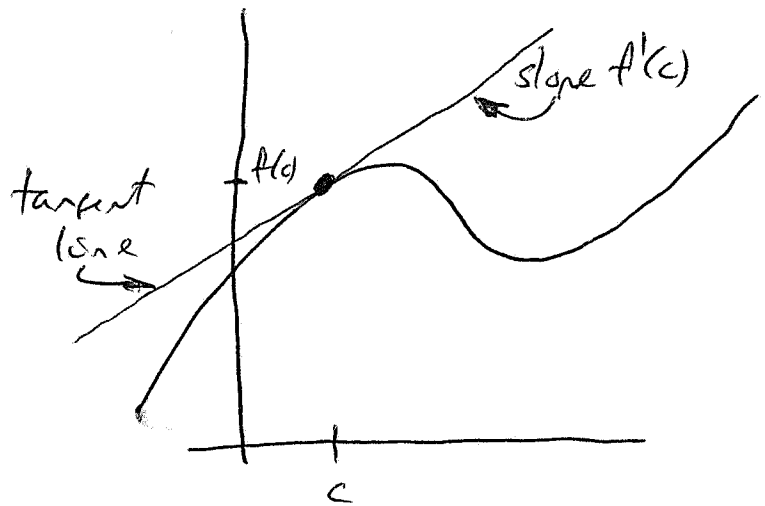
- $f'(x) = \frac{df}{dx} = \frac{d}{dx}(f(x))$, where $\lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx}$

- For $y = f(x)$, $y'(x) = \frac{dy}{dx}$

- $f'(c) = \left. \frac{df}{dx} \right|_{x=c}$

⑤ Given $y = f(x)$, if $f'(c)$ exists at $x = c$, then the ~~slope~~ ~~of the~~ line passing through $(c, f(c))$ with slope $f'(c)$ is the tangent line to the graph of $f(x)$ at $x = c$:

Recall, the line with slope m passing through (x_0, y_0) has eqn:
 $y - y_0 = m(x - x_0)$.



The eqn of the tangent line to graph of $f(x)$ at $x = c$ is then:

$$y - f(c) = f'(c)(x - c)$$

or

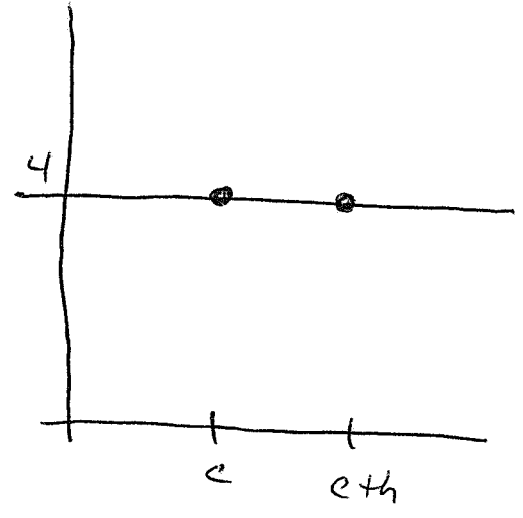
$$y = f'(c)(x - c) + f(c)$$

Q: Must f be continuous at $x = c$ for $f'(c)$ to exist? why or why not?

Here, the definition of derivative can be used to calculate:

ex. let $f(x) = 4$.

Here, for any choice of $c \in \mathbb{R}$, we have



$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{4 - 4}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0.$$

Hence $f(x) = 4$ is differentiable at every point in its domain, and its derivative is always 0. (The tangent line to the graph is the graph itself.)

ex. Find $f'(c)$ for any choice of c where $f(x) = ax + b$.

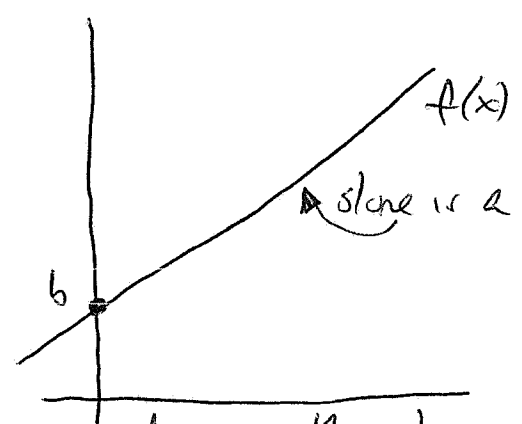
Here for any choice of $c \in \mathbb{R}$,

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{a(c+h) + b - (ac + b)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{ac + ah + b + b - ac - b}{h}$$

$$= \lim_{h \rightarrow 0} \frac{ah}{h} = \lim_{h \rightarrow 0} a = a.$$



Again the slope of any tangent line to $f(x)$ at $x=c$ is a .