Class 14: Section 4.1

Recall that the derivative of a function $f(x)$ at a pt $x=c$, if it exists, is

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

If we collect up all of the derivatives of f at all possible pts, we get another function.

__Def__ Given $f(x)$ differentiable on some domain, the derivative function $f'(x)$ assigns to each input x the output value $f'(x)$, when it exists. The domain of $f'(x)$ is the set of all pts in the domain of f where $f'(x)$ exists.

ex. Calculate $f'(x)$ for $f(x) = 3x^2 + 6$.

__Strategy__: Use the formal definition, leaving x as the unknown input variable

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Solution: For $f(x) = 3x^2 + 6$, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 + 6 - (3x^2+6)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 + \cancel{6} - \cancel{3x^2} - \cancel{6}}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(6x+3h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 6x + 3h \underset{\text{for limits}}{\overset{\text{Sum Rule}}{=}} \lim_{h \to 0} 6x + \lim_{h \to 0} 3h$$

$$= 6x + 0 = 6x.$$

Here $f'(x) = 6x$. ∎

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ex. Find $\frac{d}{dx}\left[\frac{1}{x}\right]$.

Strategy: Same as previous problem, thinking of $g(x) = \frac{1}{x}$, and problem asks for $g'(x)$.

Solution: $$\frac{d}{dx}\left[\frac{1}{x}\right] = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

(If $g(x) = \frac{1}{x}$, then $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$).

Solution (cont'd)

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x}{(x+h)x} - \frac{x+h}{(x+h)x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x-x-h}{(x+h)x}}{h} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(x+h)x}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h)x}$$

The domain of $g(x) = \frac{1}{x}$ is all $x \neq 0$. Hence the derivative already does not include this pt in the domain. For any other choice of x, the function $\frac{-1}{(x+h)x}$ is continuous at $h=0$.

Hence we can just plug in to get

$$= \lim_{h\to 0} \frac{-1}{(x+h)x} = \boxed{\frac{-1}{x^2} = \frac{d}{dx}\left[\frac{1}{x}\right].}$$

Q: What if the function has a corner?

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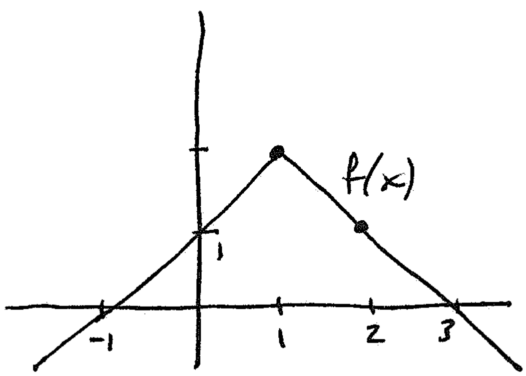
ex. Calculate $f'(x)$, for $f(x) = 2 - |x-1|$

Strategy: Rewrite the function as a piecewise defined function, and look for places where the derivative may not be defined.

Solution: Rewrite $f(x) = 2 - |x-1| = \begin{cases} x+1 & x<1 \\ -x+3 & x \geq 1 \end{cases}$



By previous example of a linear function, $f'(x) = 1$ for $x < 1$ and $f'(x) = -1$ for $x \geq 1$.

At $x = -1$, we calculate limit:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{2 - |(1+h)-1| - (2 - |1-1|)}{h}$$

$$= \lim_{h\to 0} \frac{-|h|}{h}$$. Hard to see, but

$\lim_{h\to 0^+} \frac{-|h|}{h} = \lim_{h\to 0^+} \frac{-h}{h} = -1$, and

$\lim_{h\to 0^-} \frac{-|h|}{h} = \lim_{h\to 0^-} \frac{-(-h)}{h} = 1$. Since these are not equal,

$f'(1) = \lim_{h\to 0} \frac{-|h|}{h}$ does not exist. Hence

$$f'(x) = \begin{cases} 1 & x<1 \\ -1 & x>1 \end{cases}.$$

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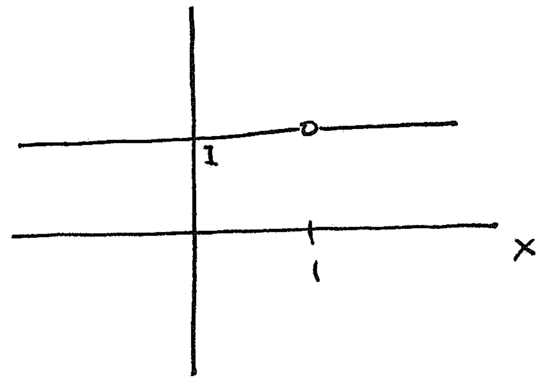
Q: What about functions with a hole at $x=c$?

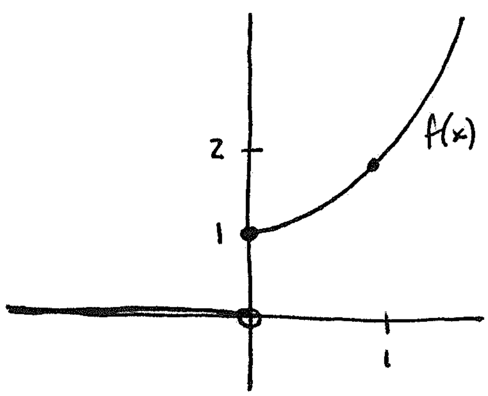
ex. Let $g(x) = \frac{x-1}{x-1}$. Calculate $g'(1)$ if it exists.



Solution: Here
$$g'(1) = \lim_{h\to 0} \frac{g(1+h)-g(1)}{h}.$$
But $g(1)$ doesn't exist.
Hence $g'(1)$ doesn't exist.

Q: What about a jump discontinuity?

ex Show $f'(0)$ does not exist when $f(x) = \begin{cases} 0 & x < 0 \\ x^2+1 & x \geq 0 \end{cases}$



Strategy: We calculate the side limits to see if the full derivative limit exists.

Solution: $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(0+h)^2+1-(0^2+1)}{h}$

$= \lim_{h\to 0^+} \frac{h^2+1-1}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0$. This is fine.

But $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{0-(0^2+1)}{h} = \lim_{h\to 0^-} \frac{1}{h} = \infty$

2 side limits are not equal, hence $f'(0)$ does not exist.

Read carefully pages 139-141. These models will be useful later.

The formal definition of the derivative is useful. But patterns do emerge, and these patterns create rules with which the formal def. is not necessary to refer to.

Some patterns

(I) If $f(x) = a$, a constant, then $\frac{d}{dx}[f(x)] = f'(x) = 0$ for all x. Why? Since

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a-a}{h} = \lim_{h\to 0} 0 = 0.$$

(II) If $f(x) = mx+b$, then $f'(x) = m$ for all $x \in \mathbb{R}$. (see previous example).

(III) For $c \in \mathbb{R}$, a constant, $\frac{d}{dx}[cf(x)] = c f'(x)$.

This is because

$$\frac{d}{dx}[cf(x)] = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h\to 0} \frac{c(f(x+h) - f(x))}{h}$$

$$= c \underbrace{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}_{f'(x)} = c f'(x).$$

Note: Rule III says that

"The derivative of a constant times a function is the constant times the derivative of the function".

(IV) Suppose $f(x)$ and $g(x)$ are both differentiable, so $f'(x)$ and $g'(x)$ both exist. Then

$$\frac{d}{dx}[f(x)+g(x)] = \lim_{h\to 0} \frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}.$$

$$\overset{\text{Sum Rule}}{\underset{\text{for limits}}{=}} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x).$$

Called the Sum Rule for derivatives, it says "the derivative of a sum of functions is the sum of the derivatives".

The Difference Rule for Derivatives plays the same way:

$$\frac{d}{dx}[f(x)-g(x)] = f'(x) - g'(x).$$