Class 15: [redacted] Section 4.3

Patterns found in the formal definition of a derivative lead to Rules:

Cont'd.

We have already seen:

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$
for $c \in \mathbb{R}$ any constant

Sum / Difference Rule: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
whenever f, g differentiable

Here is another:

Thm Let $n \in \mathbb{N}$. Then $\frac{d}{dx}[x^n] = nx^{n-1}$

Called the Power Rule, the derivative of a power function is another power function of one less degree.

pf. Calculate directly from the formal definition:

$$\frac{d}{dx}[x^n] = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

To play with this, understand how to expand powers of binomials: for $n \in \mathbb{N}$,

$$(x+y)^n = x^n + n x^{n-1} y + \frac{n(n-1)}{2} x^{n-2} y^2 + \cdots$$
$$+ n x y^{n-1} + y^n$$

where the powers of x are descending, the powers of y are ascending, and the coefficients follow patterns of combinations: $\binom{n}{n-m} x^{n-m} h^m$, and also follow Pascal's Triangle.

1 - - - -	$(x+y)^0$
1 1 - - -	$(x+y)^1 = 1x + 1y$
1 2 1 - - -	$(x+y)^2 = 1x^2 + 2xy + 1y^2$
1 3 3 1	$\vdots$
1 4 6 4 1	
1 5 10 10 5 1	

Proof cont'd

Back to the derivative calculation: Let $n \in \mathbb{N}$,

$$\frac{d}{dx}[x^n] = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + \overbrace{\frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}^{\text{all have } h^2 \text{ or higher power}} - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left(nx^{n-1} + \overbrace{\frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}}^{\text{all still have an } h}\right)}{\cancel{h}}$$

every terms with a h in it now goes to 0. what is left is only

$$= nx^{n-1}.$$

Notes ① Actually the Power Rule works for any real number power:

$$\frac{d}{dx}[x^r] = rx^{r-1}, \quad r \in \mathbb{R}.$$

However proving this is harder.

(keep in mind, $r=0$ is a strange one. Try it: it still works).

Notes cont'd.

(2) Now all polynomials are easily differentiable:

ex: $\frac{d}{dx}[3x^6 - 4x^2 + 6x - 2] =$

Sum Rule → $\frac{d}{dx}[3x^6] + \frac{d}{dx}[-4x^2] + \frac{d}{dx}[6x] + \frac{d}{dx}[2]$

Constant Mult Rule → $= 3\frac{d}{dx}[x^6] - 4\frac{d}{dx}[x^2] + 6\frac{d}{dx}[x] + 0$

Power Rule → $= 3(6x^5) - 4(2x^1) + 6(1x^0) + 0$

$= 18x^5 - 8x + 6$

Look for the ability to do many of these steps without writing down each step.

(3) Works with negative powers:

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-1-1} = -x^{-2} = \frac{-1}{x^2}$$

And with fractional powers:

$$\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Notes cont'd.

(4) Exponential functions are not power functions:

$$\frac{d}{dx}[e^x] \neq x e^{x-1} \quad !!!$$

Q: Here is a funny problem:

Q: Is it possible to find 2 functions $f(x)$, $g(x)$ that satisfy the equation

$$x = f(x)\, g(x)$$

where $f(x)$, $g(x)$ are differentiable and

$$f(0) = g(0) = 0 ?$$

A: The answer is no, it is not possible, but the reason involves the derivative of the product $f(x)g(x)$.

Yet another Rule:

Thm Let $f(x)$, $g(x)$ be differentiable at x.

Then $$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

Note: The derivative of a product of functions is NOT the product of the derivatives. This is important.

pf. Calculate:

$$\frac{d}{dx}\left[f(x)g(x)\right] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{- f(x)g(x+h) + f(x)g(x+h)}^{\text{clever form of zero.}} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

$$\stackrel{?}{=} \underbrace{\left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right)}_{f \text{ is diff.}} \underbrace{\lim_{h\to 0} g(x+h)}_{g \text{ is cont.}} + \underbrace{\lim_{h\to 0} f(x)}_{} \underbrace{\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}_{g \text{ is diff.}}$$

$$= f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

∎

ex. Calculate $h'(x)$, for $h(x) = (2x^2)(x^3-3x+1)$

Strategy: We use the Product Rule directly, and then combine first and use the power, sum and constant multiple Rules to check.

Solution:

$$\frac{d}{dx}\left[(2x^2)(x^3-3x+1)\right] \underset{\text{Rule}}{\overset{\text{Prod}}{=}} \frac{d}{dx}\left[2x^2\right](x^3-3x+1) + (2x^2)\frac{d}{dx}\left[x^3-3x+1\right]$$

$$= (4x)(x^3-3x+1) + (2x^2)(3x^2-3)$$

$$= 4x^4-12x^2+4x+6x^4-6x^2$$

$$= \underbrace{10x^4-18x^2+4x}$$

combine first

$$\frac{d}{dx}\left[(2x^2)(x^3-3x+1)\right] = \frac{d}{dx}\left[2x^5-6x^3+2x^2\right] = \underbrace{10x^4-18x^2+4x}$$

equal

One more Rule Quotient Rule

Thm For f, g differentiable functions;

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Pf.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)g(x)}{g(x+h)g(x)} - \frac{f(x)g(x+h)}{g(x)g(x+h)}}{h.}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h\cdot g(x+h)g(x)}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) \overbrace{- f(x)g(x) + f(x)g(x)}^{\text{clever form of } 0} - f(x)g(x+h)}{h\cdot g(x+h)g(x)}$$

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x)}{h\cdot g(x+h)g(x)} - \frac{f(x)(g(x+h) - g(x))}{h\cdot g(x+h)g(x)}$$

$$\overset{?}{=} \left(\lim_{h\to 0}\frac{1}{g(x+h)g(x)}\right)\cdot\left[\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)\left(\lim_{h\to 0} g(x)\right) - \left(\lim_{h\to 0} f(x)\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\right]$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)g(x).}$$

∎

Notes

① Calculate the derivative of $\frac{1}{x^2}$ three ways:

ⓐ Power Rule:

$$\frac{d}{dx}\left[\frac{1}{x^2}\right] = \frac{d}{dx}\left[x^{-2}\right] = -2x^{-3} = \frac{-2}{x^3}.$$

ⓑ Product Rule

$$\frac{d}{dx}\left[\frac{1}{x^2}\right] = \frac{d}{dx}\left[\frac{1}{x}\cdot\frac{1}{x}\right] = \frac{d}{dx}\left[\frac{1}{x}\right]\cdot\frac{1}{x} + \frac{1}{x}\cdot\frac{d}{dx}\left[\frac{1}{x}\right]$$

$$= -\frac{1}{x^2}\cdot\frac{1}{x} + \frac{1}{x}\left(-\frac{1}{x^2}\right) = \frac{-2}{x^3}.$$

ⓒ Quotient Rule

$$\frac{d}{dx}\left[\frac{1}{x^2}\right] = \frac{\frac{d}{dx}[1]\cdot x^2 - 1\cdot\frac{d}{dx}[x^2]}{(x^2)^2}$$

$$= \frac{0\cdot x^2 - 2x}{x^4} = \frac{-2x}{x^4} = \frac{-2}{x^3}.$$

② We can now use quotient Rule to ~~calculate~~ prove the power rule for all negative integers.