

Class 16: ~~Section 4.4~~ Section 4.4 I

Another Rule: So what about compositions of functions?

$$\frac{d}{dx} [(f \circ g)(x)] = \frac{d}{dx} [f(g(x))] = ?$$

Here, if $g(x)$ is diff at $x=c$, and $f(x)$ is diff at $x=g(c)$, then $f(g(x))$ is differentiable at $x=c$.

We can "see" the pattern simply by understanding the definition:

$$\left. \frac{d}{dx} [f(g(x))] \right|_{x=c} = \lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{x - c}$$

(This is the other definition of a derivative at a pt $x=c$, from the middle of pg 134.)

$$\left. \frac{d}{dx} [f(g(x))] \right|_{x=c} = \lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{x - c} \left(\frac{g(x) - g(c)}{g(x) - g(c)} \right)$$

clever form of 1
to better understand
the structure

$$= \lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c}$$

prod rule
for
limits

$$\left(\lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \right) \cdot \left(\lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \right)$$

think of this as
a change of variables
from x to $g(x)$

$$g'(c)$$

where as $x \rightarrow c$, $g(x) \rightarrow g(c)$.

$$f'(g(c))$$

Hence $\left. \frac{d}{dx} [f(g(x))] \right|_{x=c} = f'(g(c)) \cdot g'(c)$

This is called the Chain Rule for derivatives.
in expressions.

Notes ① In essence, the derivative of a composition of functions is the product of the derivatives, but with a "twist": the outside function derivative is evaluated at the inside function.

② As a function:

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$$

Can you see the pattern? One works from the outside in.

ex Calculate $h'(x)$, where $h(x) = \sqrt{x^2+1}$

Strategy: Use the Chain Rule, with $f(x) = \sqrt{x}$, $g(x) = x^2+1$, so that $h(x) = f(g(x)) = f(x^2+1) = \sqrt{x^2+1}$.

Solution: $h'(x) = f'(g(x)) \cdot g'(x)$.

$$\text{Here } f(x) = \sqrt{x}, \quad f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(x^2+1) = \frac{1}{2\sqrt{x^2+1}}$$

$$\text{and } g'(x) = \frac{d}{dx}[x^2+1] = 2x.$$

$$\text{Hence } h'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x \quad \square$$

(No need to simplify here).

~~Ques~~

Note: Notice a pattern:

$$\frac{d}{dx}[\sqrt{f(x)}] = \frac{1}{2\sqrt{f(x)}} \cdot f'(x) \dots$$

ex. Calculate $\frac{d}{dx}\left[\frac{1}{3x^2+6}\right]$.

Solution: By Quotient Rule, we know

$$\begin{aligned} \frac{d}{dx}\left[\frac{1}{3x^2+6}\right] &= \frac{\frac{d}{dx}[1](3x^2+6) - 1 \frac{d}{dx}[3x^2+6]}{(3x^2+6)^2} = \frac{0 - 6x}{(3x^2+6)^2} \\ &= \frac{-6x}{(3x^2+6)^2} \end{aligned}$$

Solution cont'd:

By the Chain Rule, $\frac{1}{3x^2+6} = f(g(x))$, where

$$f(x) = \frac{1}{x}, \text{ and } g(x) = 3x^2+6.$$

$$\begin{aligned} \text{Thus } \frac{d}{dx} \left[\frac{1}{3x^2+6} \right] &= \frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x) \\ &= \frac{-1}{(3x^2+6)^2} \cdot 6x = \frac{-6x}{(3x^2+6)^2} \quad \square \end{aligned}$$

Notice the pattern here: For any function $f(x)$ which is differentiable,

$$\frac{d}{dx} \left[\frac{1}{f(x)} \right] = \frac{-f'(x)}{[f(x)]^2}$$

There are other patterns like this on page 162. It is not necessary to know them, but they are useful.

Example 10 pg 167 is a good example of a nested function.

ex. Calculate $\frac{d}{dx} [(\sqrt{x^2+1}+1)^2]$

Solution: First, we calculate the composition:

$$\text{Let } f(x) = x^2, \text{ and } g(x) = \sqrt{x^2+1} + 1.$$

Then the Chain Rule stipulates that

$$\begin{aligned} \frac{d}{dx} [(\sqrt{x^2+1}+1)^2] &= 2(\sqrt{x^2+1}+1) \cdot \frac{d}{dx} [\sqrt{x^2+1}+1] \\ &= 2(\sqrt{x^2+1}+1) \cdot \underbrace{\left(\frac{d}{dx} [\sqrt{x^2+1}] + \frac{d}{dx} [1] \right)}_{\text{Sum Rule}} \end{aligned}$$

We now need to apply the Chain Rule again to the part we still need to differentiate.

Let $f(x) = \sqrt{x}$, $g(x) = x^2+1$, as in the last example above. We set

$$\begin{aligned} \frac{d}{dx} [(\sqrt{x^2+1}+1)^2] &= 2(\sqrt{x^2+1}+1) \cdot \left(\frac{1}{2\sqrt{x^2+1}} \right) \cdot \frac{d}{dx} (x^2+1) \\ &= 2(\sqrt{x^2+1}+1) \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x \end{aligned}$$

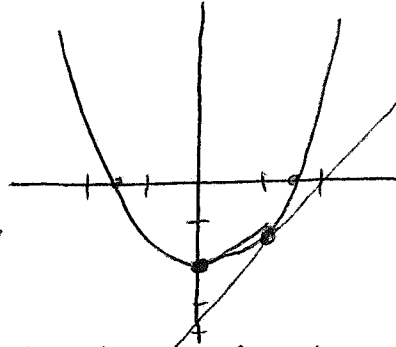
Simplify if you like, but it is not necessary.

In general, if $i(x) = f(g(h(x)))$, then

$$i'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).$$

Consider the equation relating x and y

$$y = x^2 - 2$$



Instead of thinking of y as a func of x , think instead of each side of equation as an expression involving x , where y is some unknown func of x :

Since LHS = RHS, as functions of x , their derivatives are also equal, and

Thinking of y as some unknown func of x , we can write $\frac{d}{dx}(y) = \frac{dy}{dx} \cdot \frac{dx}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx} [x^2 - 2] = 2x, \Rightarrow y' = 2x.$$

And at $x=1$, we find $y' = 2$, so $y+1 = 2(x-2)$
 $y = 2x - 5$

We can now generalize:

① Rewrite as $0 = x^2 - 2 - y$

Then $\frac{d}{dx} [0] = \frac{d}{dx} [x^2 - 2 - y] = 2x - 0 - \frac{dy}{dx}$

so again $\frac{dy}{dx} = 2x.$

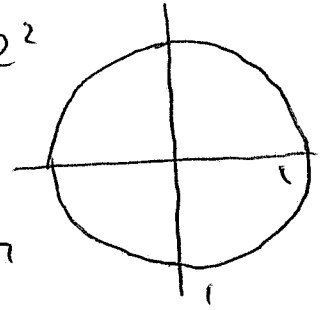
② But now, we can also do more complicated expressions:

ex. The set of all solutions to $x^2 + y^2 = 1$ is the unit circle in \mathbb{R}^2

Here, y cannot be made a

$$\text{func of } x. \quad y = \pm \sqrt{1-x^2}$$

(The northern semi-circle is $y = \sqrt{1-x^2}$)



"Implicitly", it is a function of x in the sense that its values will change as those of x change.

We can still analyse how y changes as we change x by taking derivatives of the equation (deriving implicitly):

$$\frac{d}{dx} [x^2 + y^2] = \frac{d}{dx} [1]$$

$$2x + 2y \frac{dy}{dx} = 0$$

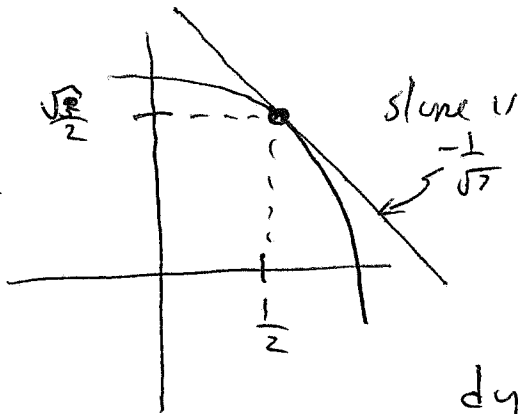
chain rule

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

Notes ① This works every where as long as expression makes sense.

② For a choice of pt (x,y), interpret $\frac{dy}{dx}$ as the slope of the line tangent to curve at (x,y).

$(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = 1$
 $\frac{1}{4} + \frac{3}{4} = 1$



Interpret $\frac{dy}{dx}$ as the slope of the line tangent to curve at (x,y).

$\frac{dy}{dx} \Big|_{\substack{x=\frac{1}{2} \\ y=\frac{\sqrt{3}}{2}}} = \frac{-(\frac{1}{2})}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$

③ This particular expression for $\frac{dy}{dx}$ works as long as $y \neq 0$. But what happens here to the tangent line?

④ Whenever an expression involving x, y cannot be written as y is a function of x, then $\frac{dy}{dx}$ can always be found but well involve both x and y. why?

(only way to denote a pt on the curve without ambiguity).