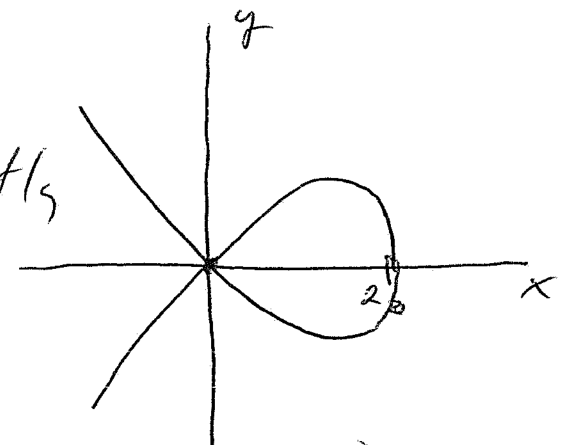
Class 17: ~~[illegible]~~ Section 4.4.

Here is another example of implicit differentiation and its utility:

ex. The equation $y^2 + x^3 = 2x^2$ is an example of a type of mathematical object called an elliptic curve (has important applications in encryption codes in information tech).

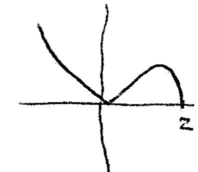
The graph of solutions to $y^2 + x^3 = 2x^2$ has no corners. However it does intersect itself. Here, neither variable can be explicitly written as a func of the other (the graph fails both the vertical and horizontal line tests.



example cont'd:

One could limit oneself to the part over the x-axis, whose equation is

$$y = \sqrt{2x^2 - x^3}$$



but this does not describe the entire curve.

We can calculate slopes of tangent lines to this curve via implicit differentiation:

$$y^2 + x^3 = 2x^2$$

$$\frac{d}{dx}\left[y^2 + x^3\right] = \frac{d}{dx}\left[2x^2\right]$$

Think of y as (some unknown) implicit fnc of x. Then since both left hand and right hand sides of equation are equal, so are their derivatives:

$$2y\frac{dy}{dx} + 3x^2 = 4x$$

The first piece is a chain rule calc on $[y(x)]^2$.

example cont'd

We can solve for $\frac{dy}{dx}$ here:

$$\frac{dy}{dx} = \frac{4x-3x^2}{2y}.$$

Notes ① Even if we cannot solve an equation for y as an explicit function of x, we can ALWAYS solve for $\frac{dy}{dx}$ ~~as an~~ via implicit differentiation.

② Again, the expression for $\frac{dy}{dx}$ will in general include both x and y (since we need both to determine where we are on the curve).

③ When $y=0$, $x=2$, the tangent line looks vertical. Can you "see" this in the expression for $\frac{dy}{dx}$. As $x \to 2$ and $y \to 0$, ~~top~~ bottom expression goes to 0 but top expression doesn't.

Notes cont'd.

④ As both x, y go to 0, the tangent lines both look like they exist (following each "leg"). Why?

The expression for $\frac{dy}{dx}$ is again undefined here, though. But this time both the top and bottom expressions go to 0 as $x, y \to 0$.

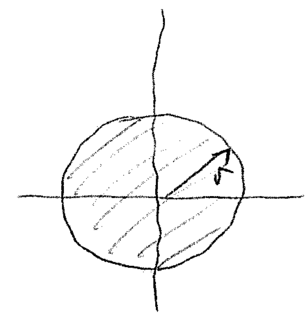
'Q: Is it possible to study the limit here and see that it does exist?

Another Application of the Chain Rule:

Related Rates

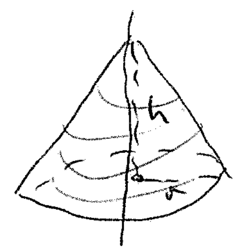
Many standard equations relating 2 variables come from geometry:

- Area of a circle: $A = \pi r^2$



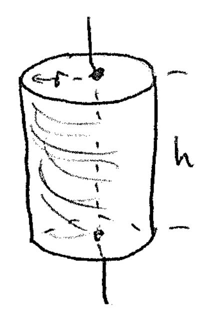
- Volume of a right circular cone:

$$V = \frac{1}{3}\pi r^2 h$$



- Surface area of a right cylinder (with top and bottom)

$$S = 2\pi r h + 2\pi r^2$$



Suppose in an application, we encounter a shape like this that is changing as time progresses....

Then at least one of the variables will change in time, forcing at least one other to change also (the equation must still hold).

ex. Drop a pebble in a still pond and watch the resulting rings of waves. Here, the radius of each ring is now a function of time $r = r(t)$. It is growing in this case.

Since the area $A = \pi r^2 = \pi (r(t))^2$, the area is also changing: $A = A(t) = \pi(r(t))^2$.

Q: How is it changing?

A: Since $A(t) = \pi(r(t))^2$, the derivatives are equal also, and $\frac{d}{dt}(A(t)) = \frac{d}{dt}\left[\pi(r(t))^2\right]$.

or $$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

We have related the rates of these quantities

Suppose the question was:

Calculate ~~the~~ how fast the area is growing if the rate of increase in the radius was 10 cm/sec when the radius was 20 cm.

The solution is simply a substitution of these quantities into the related rates eqn.

$$\left.\frac{dA}{dt}\right|_{\substack{r=20 \\ \frac{dr}{dt}=10}} = \left. 2\pi r \frac{dr}{dt} \right|_{\substack{r=20 \\ \frac{dr}{dt}=10}} = 2\pi(20)(10)$$
$$= 400\pi \ cm^2/sec$$

Notes ① It is a very good idea to ALWAYS keep units in order, and write

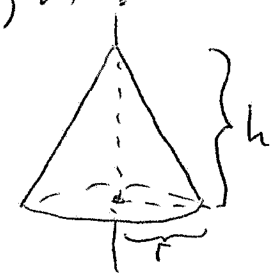
$$\left.\frac{dA}{dt}\right|_{\substack{r=20\,cm \\ \frac{dr}{dt}=10\,cm/sec}} = 2(\pi)(20\,cm)(10\,cm/sec)$$
$$= 400\pi \ cm^2/sec$$

to ensure your calculations are valid.

What if the equation contains more than 2 variables? (Most volumes are like this).

In general, all measurements may be changing when one is. But procedure is the same:

Suppose the volume of a right circular cone is changing over time. How does that affect its radius and height?



Here $V = \frac{1}{3}\pi r^2 h$, and

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{1}{3}\pi r^2 h\right]$$

$$= \frac{\pi}{3}\frac{d}{dt}\underbrace{\left[r^2 h\right]}_{\text{Product Rule}} = \frac{\pi}{3}\left[2r\frac{dr}{dt}\cdot h + r^2\frac{dh}{dt}\right]$$

ex. Suppose the radius of a right circular cone is a constant 12 inches, while ~~the~~ at a height of 10 inches, its volume is increasing at a rate of 24 inches^3/min. How fast is the height changing?

<u>Solution</u>: We evaluate the previous related rates equation at the values above in the problem:

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2 r \frac{dr}{dt} h + r^2 \frac{dh}{dt}\right]$$

$$24\ \text{in}^3/\text{min} = \frac{\pi}{3}\Big(2(12\text{ inches})(0\ \text{in}/\text{min})(10\text{ inches}) + (12\text{ inches})^2 \frac{dh}{dt}$$

The only unknown is $\frac{dh}{dt}$, and

$$\frac{dh}{dt} = \frac{24(3)}{144\pi}\ \text{in}/\text{min}.$$

The height is increasing at a rate of $\frac{1}{2\pi}$ in/min.